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CONVERGENCE OF CR-ITERATION TO COMMON FIXED POINTS OF THREE G-NONEXPANSIVE MAPPINGS IN BANACH SPACES WITH GRAPHS

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ABSTRACT

The research introduces CR-iteration process and establishes some results about the weak and strong convergence of CR-iteration process to common fixed points of three G-nonexpansive mappings in uniformly convex Banach spaces with graphs. In addition, a numerical example is provided to illustrate for the convergence of CR-iteration process to common fixed points three G-nonexpansive mappings.

Keywords: *G*-nonexpansive mapping, *CR*-iteration process, Banach space with graph.

1. Introduction

In fixed point theory, nonexpansive mappings play an important role in studying the existence and approximation of fixed points of nonlinear mappings in Banach spaces. In recent times, the nonexpansive mappings were extended and generalized in many various ways. In 2012, Aleomraninejad, Rezapour, and Shahzad introduced the notion of *G*-nonexpansive mappings in metric spaces with directed graphs and stated the convergence for this class mapping in complete metric spaces with directed graphs. In 2015, Tiammee, Kaewkhao, and Suantai proved Browder 's convergence theorem for *G*-nonexpansive mappings and studied the convergence of Halpern iteration to projecting of initial point onto the set of fixed points of *G*-nonexpansive mappings in Hilbert spaces with directed graphs. In 2016, Tripak proved the convergence of Ishikawa iteration to some common fixed points of two *G*-nonexpansive mappings in Banach spaces with directed graphs. In 2018, Kangtunyakarn generalized the results in (Tripak, 2016) to proving Halpern iteration for a finite family of *G*-nonexpansive mappings in Banach spaces with directed graphs. After that, Suparatulatorn, Cholamjiak, and Suantai generalized the results in (Kangtunyakarn, 2018) and proposed the convergence of *S*-iteration to some common fixed points of two *G*-nonexpansive mappings in Banach spaces with directed graphs. In 2018, Sridarat, Suparatulatorn, Suantai, and Cho established the convergence of *SP*-iteration to common fixed points of three *G*-nonexpansive mappings in uniformly Banach spaces with directed graphs. An interesting work naturally rises is to continue studying the convergence to common fixed points of G-nonexpansive mappings by some generalized iterations in Banach spaces with directed graphs.

In recent years, several iteration methods were proposed for approximating fixed points of nonexpansive mappings. In 2012, Chugh, Kumar, and Kumar introduced *CR* iterative scheme and studied the convergence of this iteration to fixed points of quasi-contraction in Banach spaces. In addition, the authors also showed that CR iterative scheme is faster than Picard, Mann, Ishikawa, Agarwal, Noor and SP iterative schemes by some numerical examples. However, the constructing of *CR*-type iteration process and studying the convergence of them to common fixed points of G-nonexpansive mappings in Banach spaces with directed graphs has not been considered yet. Thus, in this study, we introduce *CR*-iteration process and establish some convergence results of *CR*-iteration to common fixed points of three G-nonexpansive mappings in uniformly Banach spaces with directed graphs. First, we recall some notions and lemmas which will be useful in what follows.

Let C be a nonempty subset of a real normed space X. Let $G = (V(G), E(G))$ be a directed graph, where $V(G)$ is a set of vertices of graph G such that $V(G) = C$, $E(G)$ is a set of its edges such that $(x,x) \in E(G)$ for all $x \in C$, and G has no parallel edges.

Definition 1.1.

[(Tripak, 2016), Definition 2.4] Let X be a normed space, C be a nonempty subset of X, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then G is said to be *transitive* if for all $x,y,z \in V(G)$ such that $(x,y),(y,z) \in E(G)$, then $(x,z) \in E(G)$.

Definition 1.2.

[(Tiammee, Kaewkhao, and Suantai, 2015), p.4] Let X be a normed space, C be a nonempty subset of X, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then C is said to have the property G if for any sequence $\{x_n\}$ in C with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}^*$ and $\{x_n\}$ converges weakly to $x \in C$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $(x_{n(k)}, x) \in E(G)$ for all $k \in \mathbb{N}^*$.

Definition 1.3.

[(Shahzad & Al-Dubiban, 2006), p.534] Let X be a normed space, C be a nonempty subset of X, and $T : C \to C$ be a mapping. Then T is called *semicompact* if for any bounded sequence $\{x_n\}$ in C with $\lim_{n \to \infty} \| x_n - Tx_n \| = 0$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\{x_{n(k)}\}$ converges to $x \in C$.

Definition 1.4.

[(Tiammee et al., 2015), p.2] Let X be a normed space, C be a nonempty subset of X, $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, and $T : C \to C$ be a mapping. Then T is called *G-nonexpansive* if the following conditions hold:

(1) T is edge-preserving, that is, for all $(x, y) \in E(G)$, we have $(Tx, Ty) \in E(G)$.

(2) $\|Tx - Ty\| \le \|x - y\|$ for all $(x, y) \in E(G)$.

We denote that $F(T) = \{x \in C : Tx = x\}$ is the set of fixed points of a mapping $T : C \to C$. The following result shows that the sufficient condition for the closed convex property of the set $F(T)$ with T is a G-nonexpansive mapping.

Proposition 1.5.

[(Tiammee et al., 2015), Theorem 3.2] *Let X be a normed space, C be a nonempty subset of X, $G = (V(G), E(G))$ be a directed graph with $V(G) = C$, $E(G)$ being convex, C have the property G, and $T : C \to C$ be a G-nonexpansive mapping such that $F(T) \times F(T) \subset E(G)$. Then $F(T)$ is closed and convex.*

Definition 1.6.

[(Dozo, 1973), Definition 1.1] Let X be a normed space. Then X is said to satisfy *Opial's condition* if for any $x \in X$ and $\{x_n\}$ converges weakly to x, we have

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \text{ for all } y \in X, y \neq x.$$

Proposition 1.7.

[(Sridarat, Suparatulatorn, Suantai, and Cho, 2018), Proposition 3.5] *Let X be a Banach space satisfying the Opial's condition, C be a nonempty subset of X, $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, C have the property G, $T : C \to C$ be a G-nonexpansive, $\{x_n\}$ be a sequence in C such that $\{x_n\}$ converges weakly to $p \in C, (x_n, x_{n+1}) \in E(G)$ and $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. Then $Tp = p$.*

Definition 1.8.

[(Sridarat et al., 2018), p.13] Let X be a normed space, C be a nonempty subset of X and $T_1, T_2, T_3 : C \to C$ be three mappings. Then T_1, T_2, T_3 are said to satisfy the *condition (C)* if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r > 0$ and for all $x > 0$ such that

$$\max\{\|x - T_1 x\|, \|x - T_2 x\|, \|x - T_3 x\|\} \ge f(d(x, F)),$$

where $F = F(T_1) \cap F(T_2) \cap F(T_3)$ and $d(x, F) = \inf\{\|x - y\| : y \in F\}$.

Lemma 1.9.

[(Schu, 1991), Lemma 1.3] *Let X be a uniformly convex Banach space,* $\{\alpha_n\}$ *be a sequence in* $[\delta, 1-\delta]$ *with* $\delta \in (0,1)$ *and* $\{x_n\}, \{y_n\}$ *be two sequences in X such that*

$$\limsup_{n\to\infty} \|x_n\| \le r, \limsup_{n\to\infty} \|y_n\| \le r \text{ and } \lim_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n) y_n\| = r \text{ with } r \ge 0.$$

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0.$

2. Main results

By combining *CR* iterative scheme in (Chugh, Kumar, and Kumar, 2012) with *SP*-iteration process in (Sridarat et al., 2018), we introduce *CR*-iteration process $\{x_n\}$ for three *G*-nonexpansive mappings as follows:

$$x_1 \in C, \begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T_1 x_n \\ y_n = (1-\beta_n)T_1 x_n + \beta_n T_2 z_n \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T_3 y_n \end{cases} \text{for all } n \in \mathbb{N}^*, \tag{2.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$, C is a nonempty closed convex subset of a Banach space X and $T_1, T_2, T_3 : C \to C$ are three *G*-nonexpansive mappings.

Next, we establish some properties of *CR*-iteration.

Proposition 2.1.

Let X be a normed space, C be a nonempty convex subset of X, $G = (V(G), E(G))$ *be a directed graph which is transitive with* $V(G) = C, E(G)$ *being convex,* $T_1, T_2, T_3 : C \to C$ *be three G-nonexpansive mappings* $, \{x_n\}$ *be a sequence defined by recursion (2.1) satisfying* $(x_1, p), (p, x_1) \in E(G)$ *with* $p \in F$. *Then* $(x_n, p), (y_n, p), (z_n, p)$, $(p, x_n), (p, y_n)$, $(p, z_n), (x_n, y_n), (x_n, z_n), (x_n, x_{n+1}) \in E(G)$ *for all* $n \in \mathbb{N}^*$.

Proof.

We will prove $(p, x_n), (p, y_n), (p, z_n) \in E(G)$ by using mathematical introduction. First, we prove that $(p, y_1), (p, z_1) \in E(G)$. Indeed, since $p \in F$, we have $p \in F(T_i)$ and hence $T_i p = p$ for all $i = \overline{1,3}$. Since $(p, x_1) \in E(G)$ and T_1 is edge-preserving, we get $(p, T_1 x_1) \in E(G)$. Moreover,

$$(p, z_1) = (p, (1-\gamma_1)x_1 + \gamma_1 T_1 x_1) = (1-\gamma_1)(p, x_1) + \gamma_1 (p, T_1 x_1). \tag{2.2}$$

Since (p, x_1) and $(p, T_1 x_1) \in E(G)$, from (2.2), we get $(p, z_1) \in E(G)$. Combining this with the edge-preserving property of T_2, we obtain $(p, T_2 z_1) \in E(G)$. Furthermore,

$$(p, y_1) = (p, (1-\beta_1)T_1 x_1 + \beta_1 T_2 z_1) = (1-\beta_1)(p, T_1 x_1) + \beta_1 (p, T_2 z_1). \tag{2.3}$$

Then, combining (2.3) with (p,T_1x_1) and $(p,T_2z_1) \in E(G)$, we get that $(p,y_1) \in E(G)$.

Next, suppose that $(p,x_k) \in E(G)$ for all $k \geq 1$. We will prove that $(p,x_{k+1}),(p,y_{k+1})$, $(p,z_{k+1}) \in E(G)$. Indeed, since T_1 is edge-preserving, we obtain $(p,T_1x_k) \in E(G)$. Moreover,

$$(p,z_k) = (p,(1-\gamma_k)x_k + \gamma_k T_1x_k) = (1-\gamma_k)(p,x_k) + \gamma_k(p,T_1x_k). \tag{2.4}$$

Thus, combining (2.4) with (p,x_k) and $(p,T_1x_k) \in E(G)$, we have $(p,z_k) \in E(G)$. Then, since T_2 is edge-preserving, we obtain $(p,T_2z_k) \in E(G)$. Furthermore,

$$(p,y_k) = (p,(1-\beta_k)T_1x_k + \beta_k T_2z_k) = (1-\beta_k)(p,T_1x_k) + \beta_k(p,T_2z_k). \tag{2.5}$$

By combining (2.5) with (p,T_1x_k) and $(p,T_2z_k) \in E(G)$, we obtain $(p,y_k) \in E(G)$. Since T_3 is edge-preserving, we get $(p,T_3y_k) \in E(G)$. We also have

$$(p,x_{k+1}) = (p,(1-\alpha_k)y_k + \alpha_k T_3y_k) = (1-\alpha_k)(p,y_k) + \alpha_k(p,T_3y_k). \tag{2.6}$$

Thus, from $(p,y_k),(p,T_3y_k) \in E(G)$ and (2.6), we obtain $(p,x_{k+1}) \in E(G)$. Combining this with the edge-preserving property of T_1, we conclude that $(p,T_1x_{k+1}) \in E(G)$. Furthermore,

$$(p,z_{k+1}) = (p,(1-\gamma_{k+1})x_{k+1} + \gamma_{k+1}T_1x_{k+1}) = (1-\gamma_{k+1})(p,x_{k+1}) + \gamma_{k+1}(p,T_1x_{k+1}). \tag{2.7}$$

It follows from $(p,x_{k+1}),(p,T_1x_{k+1}) \in E(G)$ and (2.7), we obtain $(p,z_{k+1}) \in E(G)$. Since T_2 is edge-preserving, we get $(p,T_2z_{k+1}) \in E(G)$. Moreover,

$$(p,y_{k+1}) = (p,(1-\beta_{k+1})T_1x_{k+1} + \beta_{k+1}T_2z_{k+1}) = (1-\beta_{k+1})(p,T_1x_{k+1}) + \beta_{k+1}(p,T_2z_{k+1}). \tag{2.8}$$

Then, by combining (2.8) with (p,T_1x_{k+1}) and $(p,T_2z_{k+1}) \in E(G)$, we obtain $(p,y_{k+1}) \in E(G)$. Therefore, by induction, we conclude that $(p,x_n),(p,y_n),(p,z_n) \in E(G)$ for all $n \in \mathbb{N}^*$. Next, by using similar arguments as in the above proofs, we also see that $(x_n,p),(y_n,p),\ (z_n,p) \in E(G)$ for all $n \in \mathbb{N}^*$.

Finally, we shall prove that $(x_n,y_n),(x_n,z_n),(x_n,x_{n+1}) \in E(G)$. In fact, by using the transitive property of G and $(x_n,p),(p,y_n),(x_n,p),(p,z_n),(x_n,p),(p,x_{n+1}) \in E(G)$, we conclude that $(x_n,y_n),(x_n,z_n),(x_n,x_{n+1}) \in E(G)$. □

Proposition 2.2.

Let X be a normed space, C be a nonempty closed convex subset of X, $G = (V(G), E(G))$ be a directed graph which is transitive with $V(G) = C, E(G)$ being convex, $T_1, T_2, T_3 : C \to C$ be three G-nonexpansive mappings such that $F \neq \emptyset$, $\{x_n\}$ be a sequence defined by recursion (2.1) satisfying $(x_1, p), (p, x_1) \in E(G)$ with $p \in F$. Then $\{x_n\}$ is bounded and $\lim_{n\to\infty} \| x_n - p \|$ exists.

Proof.

It follows from $p \in F, (p, x_1), (x_1, p) \in E(G)$ and Proposition 2.1, we conclude that $(x_n, p), (y_n, p), (z_n, p) \in E(G)$. Since $(x_n, p) \in E(G), T_1 p = p$ and T_1 is a G-nonexpansive mapping, we have

$$\begin{aligned} \| z_n - p \| &\le (1-\gamma_n) \| x_n - p \| + \gamma_n \| T_1 x_n - T_1 p \| \\ &\le (1-\gamma_n) \| x_n - p \| + \gamma_n \| x_n - p \| \\ &= \| x_n - p \| . \end{aligned} \tag{2.9}$$

Since $(x_n, p), (z_n, p) \in E(G), T_1 p = T_2 p = p$ and T_1, T_2 are two G-nonexpansive mappings, by using (2.9), we obtain

$$\begin{aligned} \| y_n - p \| &\le (1-\beta_n) \| T_1 x_n - T_1 p \| + \beta_n \| T_2 z_n - T_2 p \| \\ &\le (1-\beta_n) \| x_n - p \| + \beta_n \| z_n - p \| \\ &\le (1-\beta_n) \| x_n - p \| + \beta_n \| x_n - p \| \\ &= \| x_n - p \| . \end{aligned} \tag{2.10}$$

Since $(y_n, p) \in E(G), T_3 p = p$ and T_3 is a G-nonexpansive mapping, by using (2.10), we get

$$\begin{aligned} \| x_{n+1} - p \| &\le (1-\alpha_n) \| y_n - p \| + \alpha_n \| T_3 y_n - T_3 p \| \\ &\le (1-\alpha_n) \| y_n - p \| + \alpha_n \| y_n - p \| \\ &\le \| y_n - p \| \\ &\le \| x_n - p \| . \end{aligned}$$

(2.11)

It follows from (2.11), we conclude that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \| x_n - p \|$ exists. □

Proposition 2.3.

Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subset[\varepsilon,1-\varepsilon]$ with $\varepsilon\in(0,1)$, $G=(V(G),E(G))$ be a directed graph which is transitive with $V(G)=C, E(G)$ being convex, $T_1,T_2,T_3:C\to C$ be three G-nonexpansive mappings such that $F\neq\emptyset$, $\{x_n\}$ be a sequence defined by recursion (2.1) satisfying $(p,x_1),(x_1,p)\in E(G)$ with $p\in F$. Then

$$\lim_{n\to\infty}\|x_n-T_1x_n\|=\lim_{n\to\infty}\|x_n-T_2x_n\|=\lim_{n\to\infty}\|x_n-T_3x_n\|=0.$$

Proof.

It follows from $p\in F,(p,x_1),(x_1,p)\in E(G)$ and Proposition 2.1, we conclude that $(p,x_n),(x_n,p),(y_n,p),(z_n,p),(z_n,x_n),\ (y_n,x_n)\in E(G)$ for all $n\in\mathbb{N}^*$.

Furthermore, from Proposition 2.2, we get that $\lim_{n\to\infty}\|x_n-p\|$ exists. Put $\lim_{n\to\infty}\|x_n-p\|=c$. Then, from (2.9), we obtain

$$\limsup_{n\to\infty}\|z_n-p\|\le\limsup_{n\to\infty}\|x_n-p\|=c. \tag{2.12}$$

Since $(x_n,p)\in E(G)$ and T_1 is a G-nonexpansive mapping, we obtain

$$\|T_1x_n-p\|\le\|T_1x_n-T_1p\|\le\|x_n-p\|.$$

Therefore,

$$\limsup_{n\to\infty}\|T_1x_n-p\|\le\limsup_{n\to\infty}\|x_n-p\|=c. \tag{2.13}$$

Since $(x_n,p),(y_n,p),(z_n,p)\in E(G)$, and T_1,T_2 are two G-nonexpansive mappings, by (2.10) and (2.11), we conclude that

$$\|x_{n+1}-p\|\le\|y_n-p\|\le(1-\beta_n)\|x_n-p\|+\beta_n\|z_n-p\|.$$

This implies that $\beta_n\|x_n-p\|\le\|x_n-p\|-\|x_{n+1}-p\|+\beta_n\|z_n-p\|$. Therefore,

$$\|x_n-p\|\le\frac{1}{\beta_n}\big(\|x_n-p\|-\|x_{n+1}-p\|\big)+\|z_n-p\|.$$

$$\le\frac{1}{\varepsilon}\big(\|x_n-p\|-\|x_{n+1}-p\|\big)+\|z_n-p\|.$$

By combining this with $\liminf_{n\to\infty}(\|x_n-p\|-\|x_{n+1}-p\|)=0$, we conclude that $c\le\liminf_{n\to\infty}\|z_n-p\|$. It follows from (2.12), we obtain $\lim_{n\to\infty}\|z_n-p\|=c$. Therefore,

$$\limsup_{n\to\infty}\|(1-\gamma_n)(x_n-p)+\gamma_n(T_1x_n-p)\|=\limsup_{n\to\infty}\|z_n-p\|=c.$$

By combining this with $\limsup_{n\to\infty}\|T_1x_n - p\| \le c, \limsup_{n\to\infty}\|x_n - p\| \le c,$ from Lemma 1.9, we obtain that

$$\lim_{n\to\infty}\|T_1x_n - x_n\| = 0. \tag{2.14}$$

Furthermore, since $\|z_n - x_n\| = \gamma_n\|T_1x_n - x_n\|,$ by (2.14), we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{2.15}$$

Next, from (2.10), we see that

$$\limsup_{n\to\infty}\|y_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| = c. \tag{2.16}$$

Since $(x_n, p) \in E(G)$ and T_2 is a G-nonexpansive, we get

$$\|T_2z_n - p\| = \|T_2z_n - T_2p\| \le \|z_n - p\| \le \|x_n - p\|.$$

This implies that $\limsup_{n\to\infty}\|T_2z_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| = c$. Moreover, by (2.11), we have

$$\liminf_{n\to\infty}\|x_{n+1} - p\| \le \liminf_{n\to\infty}\|y_n - p\|.$$

By combining the above with $\liminf_{n\to\infty}\|x_{n+1} - p\| = c,$ we have $c \le \liminf_{n\to\infty}\|y_n - p\|$. By combining this with (2.16), we conclude that $\lim_{n\to\infty}\|y_n - p\| = c.$ Thus,

$$\limsup_{n\to\infty}\|(1-\beta_n)(T_1x_n - p) + \beta_n(T_2z_n - p)\| = \limsup_{n\to\infty}\|y_n - p\| = c.$$

By combining this with $\limsup_{n\to\infty}\|T_1x_n - p\| \le c, \limsup_{n\to\infty}\|T_2z_n - p\| \le c,$ by Lemma 1.9, we conclude that

$$\lim_{n\to\infty}\|T_1x_n - T_2z_n\| = 0. \tag{2.17}$$

Since $(z_n, x_n) \in E(G)$ and T_2 is a G-nonexpansive mapping, we see that

$$\begin{aligned}\|x_n - T_2x_n\| &\le \|x_n - T_1x_n\| + \|T_1x_n - T_2z_n\| + \|T_2z_n - T_2x_n\| \\ &\le \|x_n - T_1x_n\| + \|T_1x_n - T_2z_n\| + \|z_n - x_n\|.\end{aligned}$$

By the above inequality, (2.14), (2.15) and (2.17), we conclude that $\lim_{n\to\infty}\|x_n - T_2x_n\| = 0.$ Furthermore, from $\|y_n - T_1x_n\| = \beta_n\|T_2z_n - T_1x_n\|$ and (2.17), we obtain

$$\lim_{n\to\infty}\|y_n - T_1x_n\| = 0. \tag{2.18}$$

Then, from $\|y_n - x_n\| \le \|y_n - T_1x_n\| + \|T_1x_n - x_n\|,$ (2.14) and (2.18), we have

$$\lim_{n\to\infty} \| y_n - x_n \| = 0. \tag{2.19}$$

Furthermore, by $(y_n, p) \in E(G)$ and T_3 is a *G*-nonexpansive, we get

$\| T_3 y_n - p \| \le \| T_3 y_n - T_3 p \| \le \| y_n - p \| \le \| x_n - p \|$.

This implies that $\limsup_{n\to\infty} \| T_3 y_n - p \| \le \limsup_{n\to\infty} \| x_n - p \| = c$. Then, by Lemma 1.9 and using the following inequality: $\limsup_{n\to\infty} \| T_3 y_n - p \| \le c, \limsup_{n\to\infty} \| y_n - p \| \le c$ and

$\limsup_{n\to\infty} \| (1-\alpha_n)(y_n - p) + \alpha_n (T_3 y_n - p) \| = \limsup_{n\to\infty} \| x_{n+1} - p \| = c,$

we conclude that

$$\lim_{n\to\infty} \| T_3 y_n - y_n \| = 0. \tag{2.20}$$

We also have $\| x_{n+1} - x_n \| = \| (1-\alpha_n) y_n + \alpha_n T_3 y_n - x_n \| \le \| y_n - x_n \| + \alpha_n \| T_3 y_n - y_n \|$. This implies that $\| x_{n+1} - x_n \| \le \| y_n - x_n \| + (1-\varepsilon) \| T_3 y_n - y_n \|$. Therefore, from the above inequality, (2.19) and (2.20), we obtain

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \tag{2.21}$$

Then, by the following inequality $\| x_{n+1} - T_3 y_n \| \le \| x_{n+1} - x_n \| + \| x_n - y_n \| + \| y_n - T_3 y_n \|$, (2.19), (2.20) and (2.21), we have

$$\lim_{n\to\infty} \| x_{n+1} - T_3 y_n \| = 0. \tag{2.22}$$

Since $(y_n, x_n) \in E(G)$ and T_3 is a *G*-nonexpansive, we get

$$\begin{aligned} \| x_n - T_3 x_n \| &\le \| x_n - x_{n+1} \| + \| x_{n+1} - T_3 y_n \| + \| T_3 y_n - T_3 x_n \| \\ &\le \| x_n - x_{n+1} \| + \| x_{n+1} - T_3 y_n \| + \| y_n - x_n \|. \end{aligned} \tag{2.23}$$

Therefore, by (2.19), (2.21), (2.22), (2.23), we conclude that $\lim_{n\to\infty} \| x_n - T_3 x_n \| = 0$ □

The following result is a sufficient condition for the weak convergence of iteration process (2.1) to common fixed points of three *G*-nonexpansive mappings in uniformly convex Banach spaces.

Theorem 2.4.

Let X be a uniformly convex Banach space satisfying the Opial's condition, C be a nonempty closed convex subset of X, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [\varepsilon, 1-\varepsilon]$ *with* $\varepsilon \in (0,1)$, $G = (V(G), E(G))$ *be a directed graph which is transitive with* $V(G) = C, E(G)$ *being convex, C have the property G,* $T_1, T_2, T_3 : C \to C$ *be three G-nonexpansive mappings such* *that* $F \neq \emptyset$, $\{x_n\}$ *be a sequence defined by recursion (2.1) satisfying* $(p,x_1),(x_1,p) \in E(G)$ *with* $p \in F$. *Then* $\{x_n\}$ *converges weakly to* $q \in F$.

Proof.

Since X is a uniformly convex Banach space, we see that X is a reflexive Banach space. Moreover, by Proposition 2.2, we get that $\{x_n\}$ is bounded. Therefore, there exists a subsequence $\{x_{n(i)}\}$ of $\{x_n\}$ such that $\{x_{n(i)}\}$ converges weakly to $q \in C$. Then, by Proposition 2.3, we conclude that

$$\lim_{i\to\infty} \| x_{n(i)} - T_1 x_{n(i)} \| = \lim_{i\to\infty} \| x_{n(i)} - T_2 x_{n(i)} \| = \lim_{i\to\infty} \| x_{n(i)} - T_3 x_{n(i)} \| = 0.$$

Thus, from the above and by Proposition 1.7, we conclude that $T_1 q = T_2 q = T_3 q = q$ and hence $q \in F = F(T_1) \cap F(T_2) \cap F(T_3)$.

Suppose that $\{x_n\}$ does not converges weakly to q. Then, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\{x_{n(k)}\}$ converges weakly to $q_1 \in C$ with $q \neq q_1$. By using similar arguments as in the above proofs, from Proposition 1.7, we conclude that $q_1 \in F$. Furthermore, from Proposition 2.2, we get that $\lim_{n\to\infty} \| x_n - q \|$ and $\lim_{n\to\infty} \| x_n - q_1 \|$ exist. Then, by the Opial's condition, we see that

$$\lim_{n\to\infty} \| x_n - q \| = \liminf_{j\to\infty} \| x_{n(j)} - q \| < \lim_{j\to\infty} \| x_{n(j)} - q_1 \| = \lim_{n\to\infty} \| x_n - q_1 \|$$

$$= \liminf_{k\to\infty} \| x_{n(k)} - q_1 \| < \lim_{k\to\infty} \| x_{n(k)} - q \| = \lim_{n\to\infty} \| x_n - q \|,$$

which is a contradiction. This implies that $\{x_n\}$ converges weakly to $q \in F$. □

Next, we prove some strong convergence results of iteration (2.1) to common fixed points of three *G*-nonexpansive mappings in uniformly convex Banach spaces.

Theorem 2.5.

Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset [\varepsilon, 1-\varepsilon]$ *with* $\varepsilon \in (0,1)$, $G = (V(G), E(G))$ *be a directed graph which is transitive with* $V(G) = C, E(G)$ *being convex, C have the property G,* $T_1, T_2, T_3 : C \to C$ *be three G-nonexpansive mappings such that* $F \neq \emptyset$, $F(T_i) \times F(T_i) \subset E(G)$ *for all* $i = 1,2,3$ *and satisfying the condition (C),* $\{x_n\}$ *be a sequence defined by recursion (2.1) such that* $(p,x_1),(x_1,p) \in E(G)$ *with* $p \in F$. *Then* $\{x_n\}$ *converges strongly to* $q \in F$.

Proof.

It follows from $p \in F, (p, x_1), (x_1, p) \in E(G)$ and Proposition 2.1, we conclude that $(x_n, p), (y_n, p), (z_n, p) \in E(G)$ for all $n \in \mathbb{N}^*$.

Then, by Proposition 2.2, we see that $\lim_{n\to\infty} \| x_n - p \|$ exists and $\{x_n\}$ is bounded. On the other hand, by (2.11), we obtain $\| x_{n+1} - p \| \le \| x_n - p \|$ for all $n \in \mathbb{N}^*$. This implies that $d(x_{n+1}, F) \le d(x_n, F)$ and hence $\lim_{n\to\infty} d(x_n, F)$ exists. Next, by Proposition 2.3, we have

$$\lim_{n\to\infty} \| x_n - T_1 x_n \| = \lim_{n\to\infty} \| x_n - T_2 x_n \| = \lim_{n\to\infty} \| x_n - T_3 x_n \| = 0. \tag{2.24}$$

Since T_1, T_2, T_3 satisfy the condition (C), there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r > 0$ and

$$\max\{\| x_n - T_1 x_n \|, \| x_n - T_2 x_n \|, \| x_n - T_3 x_n \|\} \ge f(d(x_n, F)). \tag{2.25}$$

By combining (2.24) with (2.25), we obtain $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Suppose that $\lim_{n\to\infty} d(x_n, F) > 0$. Then for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}^*$ such that for all $n \ge n_0$, we have $d(x_n, F) > \varepsilon$. This implies that $f(d(x_n, F)) \ge f(\varepsilon)$ for all $n \ge n_0$. Therefore, $\lim_{n\to\infty} f(d(x_n, F)) \ge f(\varepsilon) \ge f(0) = 0$, which contradicts to $\lim_{n\to\infty} f(d(x_n, F)) = 0$. So $\lim_{n\to\infty} d(x_n, F) = 0$. Then, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ and a sequence $\{p_k\}$ in F such that $\| x_{n(k)} - p_k \| \le 2^{-k}$. Therefore, from (2.11), we conclude that $\| x_{n(k+1)} - p_k \| \le \| x_{n(k)} - p_k \| \le 2^{-k}$. This implies that

$$\| p_{k+1} - p_k \| \le \| p_{k+1} - x_{n(k+1)} \| + \| x_{n(k+1)} - p_k \| \le 2^{-(k+1)} + 2^{-k} \le 2^{-(k-1)}.$$

It follows that $\{p_k\}$ is a Cauchy sequence in F. Furthermore, by Proposition 1.5, we see that $F = F(T_1) \cap F(T_2) \cap F(T_3)$ is closed in Banach spaces. Thus, there exits $q \in F$ such that $\lim_{k\to\infty} p_k = q$. By combining this with $\| x_{n(k)} - q \| \le \| x_{n(k)} - p_k \| + \| p_k - q \| \le 2^{-k} + \| p_k - q \|$, we obtain $\lim_{k\to\infty} \| x_{n(k)} - q \| = 0$. Moreover, since $\lim_{n\to\infty} \| x_n - q \|$ exists, we conclude that $\lim_{n\to\infty} \| x_n - q \| = 0$ and hence $\{x_n\}$ strongly converges to $q \in F$. □

In Theorem 2.5, by replacing the assumption "*satisfying the condition (C) of three G-nonexpansive mappings*" by the assumption "*one of three G-nonexpansive mappings is semicompact*", we obtain the following result.

Theorem 2.6.

Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset [\varepsilon, 1-\varepsilon]$ *with* $\varepsilon \in (0,1)$, $G = (V(G), E(G))$ *be a directed graph which is transitive with* $V(G) = C, E(G)$ *being convex, C have the property G,* $T_1, T_2, T_3 : C \to C$ *be three G-nonexpansive mappings such that* $F \neq \emptyset$, $F(T_i) \times F(T_i) \subset E(G)$ *for each* $i = 1,2,3$ *and one of* T_1, T_2 *and* T_3 *is semicompact,* $\{x_n\}$ *be a sequence defined by recursion (2.1) such that* $(p, x_1), (x_1, p) \in E(G)$ *with* $p \in F$. *Then* $\{x_n\}$ *converges strongly to* $q \in F$.

Proof.

By Proposition 2.3, we obtain $\lim_{n\to\infty} \| x_n - T_i x_n \| = 0$ for each $i = 1,2,3$. By Proposition 2.2, we conclude that $\{x_n\}$ is bounded. By the semicompactness of one of T_1, T_2 and T_3, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\{x_{n(k)}\}$ converges strongly to $q \in C$. Then, by the property *G* of *C* and the transitive property of *G*, there exists a subsequence $\{x_{n(k(i))}\}$ of $\{x_{n(k)}\}$ such that $(x_{n(k(i))}, q) \in E(G)$. Therefore, for each $j = 1,2,3,$ we have

$$\| q - T_j q \| \le \| q - x_{n(k(i))} \| + \| x_{n(k(i))} - T_j x_{n(k(i))} \| + \| T_j x_{n(k(i))} - T_j q \|$$

$$\le \| q - x_{n(k(i))} \| + \| x_{n(k(i))} - T_j x_{n(k(i))} \| + \| x_{n(k(i))} - q \| .$$

This implies that $T_j q = q$ for each $j = 1,2,3$ and hence $q \in F$. As in the proof of Theorem 2.5, we conclude that $\lim_{n\to\infty} d(x_n, F)$ exists. Furthermore, from $d(x_{n(k)}, F) \le \| x_{n(k)} - q \|$, we see that $\lim_{k\to\infty} d(x_{n(k)}, F) = 0$ and hence $\lim_{n\to\infty} d(x_n, F) = 0$. By using similar arguments as in the proof of Theorem 2.5, we conclude that $\{x_n\}$ converges strongly to $q \in F$. □

Finally, we provide an example to illustrate for the convergence to common fixed points of three *G*-nonexpansive mappings by *CR*-iteration process which generated by (2.1). In addition, the example also shows that the convergence to common fixed points of given mappings by *CR*-iteration process is faster than *SP*-iteration process in (Sridarat et al., 2018).

Example 2.7.

Let $X = \mathbb{R}$ be a Banach space with norm given by $\| x \| = | x |$ for all $x \in \mathbb{R}$, $C = [0,2]$, $G = (V(G), E(G))$ be a directed graph defined by $V(G) = C$, $(x, y) \in E(G)$ if and only if $0.45 \le x, y \le 1.7$ and $x, y \in C$. Define three mappings $T_1, T_2, T_3 : C \to C$ by

$T_1 x = \sqrt{x},\ T_2 x = \frac{10}{31}\tan(x-1)+1,\ T_3 x = \frac{20}{31}\arcsin(x-1)+1$ for all $x \in C$.

Consider $\alpha_n = \frac{n+1}{5n+3}, \beta_n = \frac{n+4}{10n+7}$ and $\gamma_n = \frac{n+2}{8n+5}$ for all $n \in \mathbb{N}^*$. Then

(1) T_1, T_2, T_3 are three G-nonexpansive mappings. Indeed, for all $(x,y) \in E(G)$, we obtain $0.5 \le x, y \le 1.7$. Thus, for each $i = 1,2,3$, we get $0.5 \le T_i x, T_i y \le 1.7$ and hence $(T_i x, T_i y) \in E(G)$. This implies that T_1, T_2, T_3 are edge-preserving. Moreover, by calculating directly, we conclude that $\|T_i x - T_i y\| \le \|x - y\|$ for all $(x,y) \in E(G)$ and for each $i = 1,2,3$. Therefore, T_1, T_2, T_3 are three G-nonexpansive mappings.

(2) It is easy to see that $F = F(T_1) \cap F(T_2) \cap F(T_3) = \{1\} \ne \emptyset$. By choosing $x_1 = 1.4$, we obtain $(p, x_1), (x_1, p) \in E(G)$ for all $p \in F$.

Furthermore, other assumptions in Theorem 2.6 also are satisfied. Then, *CR*-iteration process $\{x_n\}$ generated by (2.1) which has the following form converges to common fixed point $p = 1$.

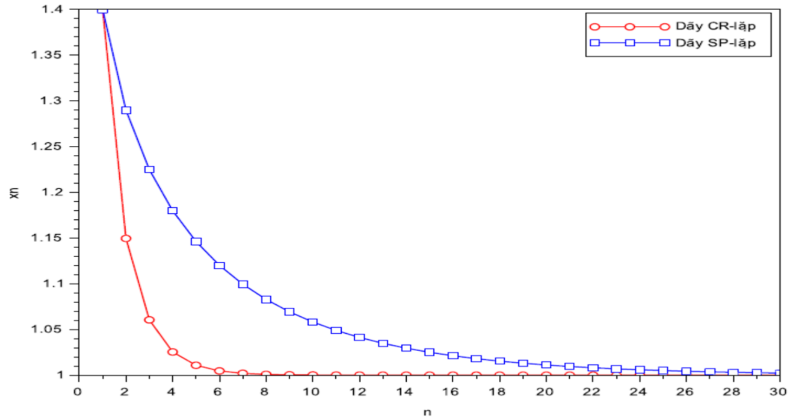
$$\begin{cases} x_1 = 1.4, \\ z_n = \frac{7n+3}{8n+5} x_n + \frac{n+2}{8n+5}\sqrt{x_n}, \\ y_n = \frac{9n+3}{10n+7}\sqrt{x_n} + \frac{n+4}{10n+7}\left(\frac{10}{31}\tan(z_n - 1) + 1\right), \\ x_{n+1} = \frac{4n+2}{5n+3} y_n + \frac{n+1}{5n+3}\left(\frac{20}{31}\arcsin(y_n - 1) + 1\right). \end{cases}$$

However, by choosing $x = 0.5, y = 0.05, u = 1.99, v = 1.96, p = 1.95$ and $q = 1.45$, we see that $|T_1 x - T_1 y| > |x - y|$, $|T_2 u - T_2 u| > |u - v|$ and $|T_3 p - T_3 q| > |p - q|$. Therefore, T_1, T_2, T_3 are not nonexpansive mappings. Thus, some convergence results to common fixed point of three nonexpansive mappings are not applicable to given mappings and the above *CR*-iteration process.

Notice that *SP*-iteration process $\{x_n\}$ was introduced in (Sridarat et al., 2018) which has following form also converges to common fixed point $p = 1$.

$$\begin{cases} x_1 = 1.4, \\ z_n = \frac{7n+3}{8n+5} x_n + \frac{n+2}{8n+5}\sqrt{x_n}, \\ y_n = \frac{9n+3}{10n+7} x_n + \frac{n+4}{10n+7}\left(\frac{10}{31}\tan(z_n - 1) + 1\right), \\ x_{n+1} = \frac{4n+2}{5n+3} y_n + \frac{n+1}{5n+3}\left(\frac{20}{31}\arcsin(y_n - 1) + 1\right). \end{cases}$$

However, the convergence of *CR*-iteration process to common fixed point $p = 1$ is faster than the convergence of *SP*-iteration process. By using Scilab-6.0.0 numerical computation software with $n = 30,$ we show the convergence behavior of *CR*-iteration process and *SP*-iteration process to common fixed point $p = 1$ as follows.



□

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SỰ HỘI TỤ CỦA DÃY CR-LẶP ĐẾN ĐIỂM BẤT ĐỘNG CHUNG CỦA BA ÁNH XẠ *G*-KHÔNG GIÃN TRONG KHÔNG GIAN BANACH VỚI ĐỒ THỊ

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TÓM TẮT

Trong bài báo này, chúng tôi giới thiệu dãy CR-lặp và thiết lập một số kết quả về sự hội tụ yếu và hội tụ của dãy CR-lặp đến điểm bất động chung của ba ánh xạ G-không giãn trong không gian Banach lồi đều với đồ thị. Đồng thời, chúng tôi cũng xây dựng ví dụ để minh họa cho sự hội tụ của dãy CR-lặp đến điểm bất động chung của ba ánh xạ G-không giãn.

Từ khóa: ánh xạ *G*-không giãn, dãy *CR*-lặp, không gian Banach với đồ thị.