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DECAY SOLUTION FOR STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION

Nguyen Nhu Quan[1], Nguyen Ngoc Vien[2]
[1]*Faculty of Natural Sciences, Electric Power University*
[2]*Faculty of Foundations, Hai Duong University*

Abstract: *In this work, we prove the existence of decay solution in mean square moment for a class of stochastic integro-differential equations with infinite delays driven by fractional Brownian motion. The existence of mild solutions is obtained by using the Banach fixed point theorem and some inequality technique.*

Keywords: *Fractional Brownian motion; Integro-differential equations; Fixed point theory; Infinite delays.*

1. INTRODUCTION

In this work, we consider the following stochastic integro-differential equation with infinite delays

$$du(t) = \left[\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)ds + f(t,u_t)\right]dt$$

$$+ g(t)dW^H(t), t>0,\ \alpha\in(1,2) \qquad (1.1)$$

$u_0 = \varphi \in \mathrm{B}$,

here the unknown function u takes values in a Banach space X, A is a sectorial operator of type (ω,θ) which will be defined below, $f:[0,\infty)\times \mathrm{B}\to X$, $g:[0,\infty)\to \mathrm{L}_2^0(Y,X)$, W^H is a fractional Brownian motion with $H\in(\frac{1}{2};1)$ on a real line and a separable Hilbert space Y, $u_t = u(t+\cdot)$ is the delay term with the interval of delay time $[-\infty,0]$, and $u(s)=\varphi(s)$ is the initial datum. Our objective is to verify the existence of mild solution to problem (1.1) with the state space B. To our knowledge, this is the first study of the existence of mild solutions to considered problem.

The fractional Brownian motion (fBm for short) is a family of centered Gaussian processes with continuous sample paths indexed by the Hurst parameter $H\in(0,1)$. It is a self-similar process with stationary increments and has a long-memory when $H>\frac{1}{2}$. These significant properties make fractional Brownian motion a natural candidate as a model for noise in a wide variety of physical phenomena, such as mathematical finance, communication networks, hydrology and medicine. For more details on fBm, we refer the readers to the articles [1, 2, 3, 8]. It is worth pointing out that, there are only a small number of results about the existence of decay solution in mean square moment to the stochastic differential equations with infinite delays. Therefore, it is necessary to develop and explore the existence criteria with these types of equations.

2. PRELIMINARIES

2.1. Resolvent operators

Let $\mathrm{L}(X)$ be the space of bounded linear operators on X. We recall some notions and results on resolvent operators related to our problem.

Definition 2.1.1. Let A be a closed and linear operator with domain $D(A)$ on a Banach space X. We say that A is the generator of an α-resolvent if there exists $\omega\in\mathbb{R}$ and a strongly continuous function $S_\alpha:\mathbb{R}^+\to \mathrm{L}(X)$ such that $\{\lambda^\alpha : \mathrm{Re}\lambda>\omega\}\subset\rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t}S_\alpha(t)xdt,$$

$$\mathrm{Re}\lambda>\omega, x\in X.$$

A case ensuring the existence of α-resolvent was discussed in [4]. Specifically, let A be a closed and densely defined operator. Assume that A is a sectorial of type (ω,θ), that is, there exist $\omega \in \mathbb{R}, \theta \in (0, \frac{\pi}{2}), M > 0$ such that its resolvent lies in $\mathbb{C} \setminus \Sigma_{\omega,\theta}$ and

$$\| (\lambda I - A)^{-1} \| \le \frac{M}{|\lambda - \omega|}, \lambda \notin \Sigma_{\omega,\theta},$$

Here $\Sigma_{\omega,\theta} = \{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\}$. In the case $0 \le \theta < \pi(1 - \alpha/2)$, $S_\alpha(\cdot)$ exists and has the following formula:

$$S_\alpha(t) = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} \lambda^{\alpha-1} (\lambda^\alpha I - A)^{-1} d\lambda, t \ge 0,$$

where γ is a suitable path lying outside $\Sigma_{\omega,\theta}$. Furthermore, we have the following assertion for the behavior of $S_\alpha(\cdot)$ which has been proved in [4, Theorem 1].

Proposition 2.1.1. Let A is a sectorial of type (ω, θ) and $0 \le \theta < \pi(1 - \alpha/2)$. Then there exists $C > 0$ independent on t such that

$$\| S_\alpha(t) \| \le \frac{C}{1 + \mu t^\alpha} \qquad (2.1)$$

for $t \ge 0$.

2.2. Phase spaces

In this work, we will deploy the axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [5].

Let $\mathcal{B}$ be a linear space of functions mapping $(-\infty, 0]$ into Bannach space X endowed with a seminorm $|\cdot|_{\mathcal{B}}$ and satisfying the following fundamental axioms. If a function $y : (-\infty, T + \sigma] \to X$ such that $y|_{[\sigma, T+\sigma]} \in C([\sigma, T + \sigma]; X)$ and $y_\sigma \in \mathcal{B}$, then

(B1) $y_t \in \mathcal{B}$ for $t \in [\sigma, T + \sigma]$;

(B2) the function $t \mapsto y_t$ is continuous on $[\sigma, T + \sigma]$;

(B3) $|y_t|_{\mathcal{B}} \le K(t - \sigma) \sup_{\sigma \le s \le t} \| y \| + M(t - \sigma) |y_\sigma|_{\mathcal{B}}$,

where $K, M : [0, \infty) \to [0, \infty)$, K is continuous, M is locally bounded, and they are independent of y.

(B4) there exists $\tilde{n} > 0$ such that $\| \varphi(0) \| \le \tilde{n} |\varphi|_{\mathcal{B}}$, for all $\varphi \in \mathcal{B}$.

We give here an examples of phase spaces. For more examples, we refer to the book by Hino, Mukarami and Naito [7].

$$C_\gamma = \{\varphi \in C((-\infty, 0]; X) : \lim_{\theta \to -\infty} e^{\gamma\theta} \varphi(\theta) \text{ exists in } X\},$$

(2.2), where γ is a positive number. This phase space satisfies (B1) - (B3) with

$$K(t) = 1, M(t) = e^{-\gamma t}, \qquad (2.3)$$

and it is a Banach space with the norm $|\varphi|_{\mathcal{B}} = \sup_{\theta \le 0} e^{\gamma\theta} \| \varphi(\theta) \|$.

2.3. Fractional Brownian motion

We first recall the definition of Wiener integrals with respect to an infinite dimensional fractional Brownian motion with Hurst index $H > \frac{1}{2}$ (see [8]).

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$ and $T > 0$ be an arbitrary fixed horizon. A one-dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (0;1)$ is a centered Gaussian process $\beta^H = \{\beta^H(t), 0 \le t \le T\}$ with the covariance function

$$R(t,s) = E[\beta^H(t) \beta^H(s)]$$
$$= \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}).$$

It is known that $\beta^H(t)$ with $H > \frac{1}{2}$ admits the following Volterra representation

$$\beta^H(t) = \int_0^t K(t,s) d\beta(s)$$

where β is a standard Brownian motion and the Volterra kernel $K(t,s)$ is given by

$$K(t,s) = C_H \int_s^t (u - s)^{H - \frac{3}{2}} \left(\frac{u}{s}\right)^{H - \frac{1}{2}} du; t \ge s.$$

For the deterministic function $\varphi \in L^2([0;T])$, the fractional Wiener integral of φ with respect to β^H is defined by

$$\int_0^T \varphi(s) d\beta^H(s) = \int_0^T K_H^* \varphi(s) d\beta(s) \quad ,$$

where $K_H^* \varphi(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r,s) dr$.

Let Y be a real, separable Hilbert space and $\mathcal{L}(Y, X)$ be the space of bounded linear operators from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $\mathcal{L}(Y, X)$. Let $\{e_n; n = 1, 2, ...\}$ be a complete orthonormal basis in Y and $Q \in \mathcal{L}(Y, Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n; n = 1, 2, ...$ are non-negative real numbers. We define the infinite dimensional fBm on X with covariance Q as

$W^H(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n\beta_n^H(t)$, where $\beta_n^H(t)$ are real, independent fBm's. This process is a X -valued Gaussian, it starts from 0, has zero mean and covariance:

$$E\langle W^H(t),x\rangle\langle W^H(s),y\rangle=R(t,s)\langle Q(x),y\rangle,$$

for all $x,y\in Y$ and $t,s\in[0;T]$.

In order to define Wiener integrals with respect to the Q-fBm $W^H(t)$, we introduce the space $\mathrm{L}_2^0:=\mathrm{L}_2^0(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi:Y\to X$. We recall that $\psi\in\mathrm{L}(Y,X)$ is called a Q -Hilbert-Schmidt operator if $\|\psi\|_{\mathrm{L}_2^0}=\sum_{n=1}^{\infty}\|\sqrt{\lambda_n}\psi e_n\|^2<\infty$ and that the space L_2^0 equipped with the inner product $\langle\varphi,\psi\rangle_{\mathrm{L}_2^0}=\sum_{n=1}^{\infty}\langle\varphi e_n,\psi e_n\rangle$ is a separable Hilbert space.

The fractional Wiener integral of the function $\psi:[0;T]\to\mathrm{L}_2^0(Y,X)$ with respect to Q-fBm is defined by

$$\begin{aligned}\int_0^t\psi(s)dW^H(s)\quad&=\sum_{n=1}^{\infty}\int_0^t\sqrt{\lambda_n}\psi(s)e_n d\beta_n^H(s)\\&=\sum_{n=1}^{\infty}\int_0^t\sqrt{\lambda_n}K_H^*(\psi e_n)(s)d\beta_n(s),\end{aligned}$$

(2.4), where β_n is the standard Brownian motion used to present β_n^H. We have the following fundamental inequality, which can be proved by similar arguments as [2, Lemma 2] and [3, Lemma 2].

Lemma 2.3.1. If $\psi:[0;T]\to\mathrm{L}_2^0(Y,X)$ satisfies $\int_0^T\|\psi(s)\|_{\mathrm{L}_2^0}^2ds<\infty$ then the sum in (2.4) is well defined as an X -valued random variable and for any $\alpha,\beta\in[0,T]$ with $\alpha>\beta$, we have

$$E\left\|\int_\alpha^\beta\psi(s)dW^H(s)\right\|^2$$
$$\le 2H(\beta-\alpha)^{2H-1}\int_\alpha^\beta\|\psi(s)\|_{\mathrm{L}_2^0}^2ds.$$

3. MAIN RESULTS

In order to study the existence of mind solutions to problem (1.1). Let $\varphi\in\mathrm{B}$, we define the space

$$\mathbf{BC}_\varphi=\{u\in C((0,+\infty);L^2(\Omega,X)):\ u(0)=\varphi(0)\},\|u\|_{BC}<\infty\},$$

clearly that it is a Banach space with the norm $\|u\|_{BC}^2=\sup_{t\ge0}\|u(t)\|^2$. We denote the set

$$B_{R,\varphi}^{\vartheta}(\rho)=\{y\in\mathbf{BC}_\varphi:\|y\|_{BC}\le R,\ \sup_{t\ge0}t^{\vartheta}E\|y(t)\|^2\le\rho\}$$

where $\vartheta=\alpha-(2H-1)>0$ and numbers $R,\rho>0$ are given. Then $B_{R,\varphi}^{\vartheta}(\rho)$ is a closed convex subset in $\mathbf{BC}_\varphi$.

For $v\in\mathbf{BC}_\varphi$, we define the function $v[\varphi]:\mathbb{R}\to X$ as follows

$$v[\varphi](t)=\begin{cases}\varphi(t), & -\infty<t\le0,\\ v(t), & t>0.\end{cases}$$

Then, clearly

$$v[\varphi]_t(\theta)=\begin{cases}\varphi(t+\theta), & -\infty<\theta<-t,\\ v(t+\theta), & \theta\in[-t,0].\end{cases}$$

In this work, we use the following definition of solution to (1).

Definition 3.1 Given $\varphi\in\mathrm{B}$. A X -valued stochastic process $\{u(t),t\in\mathbb{R}\}$ is called a mild solution of (1.1) if

(i) $u\in C\left((0,+\infty),L^2(\Omega,X)\right)$,

(ii) $u(t)=\varphi(t),t\le0$.

(iii) For arbitrary $t>0$, we have

$$u(t)=S_\alpha(t)\varphi(0)+\int_0^tS_\alpha(t-s)f(s,u_s)ds$$
$$+\int_0^tS_\alpha(t-s)g(s)dW^H(s)\quad \mathrm{P}-a.s.$$

Now we consider the operator

$$\mathrm{F}:\mathbf{BC}_\varphi\to\mathbf{BC}_\varphi,$$

$$\mathrm{F}(v)(t)=S_\alpha(t)\varphi(0)+\int_0^tS_\alpha(t-s)f(s,v_s)ds$$
$$+\int_0^tS_\alpha(t-s)g(s)dW^H(s).$$

It is clear that if v is a fixed point of F then $v[\varphi]$ is a mild solution to (1.1).

Concerning problem (1.1), we give the following assumptions:

(B) The phase space B obeys (B1) - (B4) such that the functions K and M are uniformly bounded and $\int_0^{\infty}s^{\vartheta}M^2(s)<\infty$.

(F) The map $f:\mathbb{R}^+\times\mathrm{B}\to X$ and there exists a function $\zeta\in L_{loc}^1(\mathbb{R}^+)$ such that for every $x,y\in\mathrm{B}$ we have

$\| f(t,x)-f(t,y)\|_X^2 \le \zeta(t)|x-y|_{\mathrm{B}}^2$, for a.e. $t\in\mathbb{R}^+$.

(G) The map $g:\mathbb{R}^+\to L_2^0(Y,X)$ satisfies $\int_0^\infty t^\alpha \| g(t)\|_{L_2^0}^2 dt \le \infty.$

As a consequence, we have the following result.

Lemma 3.1. Let **(F), (G)** and **(B)** hold. Then there exist two positive real numbers R and ρ such that $\mathrm{F}(B_{R,\varphi}^{\vartheta}(\rho))\subset B_{R,\varphi}^{\vartheta}(\rho)$, provide that

$$\Delta_\infty=\sup_{t\ge0}\left\{6\int_0^t\| S_\alpha(t-s)\|ds\int_0^t\| S_\alpha(t-s)\|\zeta(s)K^2(s)ds\right\}<1. \tag{3.1}$$

Proof

Here we recall that

$$\mathrm{F}(u)(t)=S_\alpha(t)\varphi(0)+\int_0^t S_\alpha(t)f(s,u_s)ds$$
$$+\int_0^t S_\alpha(t)g(s)dW^H(s),\ t\ge0$$

First, we prove that there exists a number $R>0$ such that $\mathrm{F}(B_R)\subset B_R$, where B_R is the ball in $\mathbf{BC}_\varphi$ centered at origin with radius R. Assume to the contrary that for each $n\in\mathbb{N}$, there exist $u_n\in\mathbf{BC}_\varphi$ and $z_n=\mathrm{F}(u_n)$ satisfying that $\sup_{t\ge0}E\| u_n(t)\|^2\le n$ but $\sup_{t\ge0}E\| z_n(t)\|^2>n$. From the formula of F, we have

$$E\| z_n(t)\|^2\le 3E\| S_\alpha(t)\varphi(0)\|^2$$
$$+3E\left\|\int_0^t S_\alpha(t-s)f(s,u_{ns})ds\right\|^2$$
$$+3E\left\|\int_0^t S_\alpha(t-s)g(s,u_{ns})dW^H(s)\right\|^2$$
$$=I_1(t)+I_2(t)+I_3(t),$$

where

$$I_1(t)=3E\| S_\alpha(t)\varphi(0)\|^2,$$
$$I_2(t)=3E\left\|\int_0^t S_\alpha(t-s)f(s,u_{ns})ds\right\|^2,$$
$$I_3(t)=3E\left\|\int_0^t S_\alpha(t-s)g(s,u_{ns})dW^H(s)\right\|^2.$$

Denote $S_\alpha^\infty=\sup_{t\ge0}\| S_\alpha(t)\|$, we have

$$I_1(t)\le 3(S_\alpha^\infty)^2E\|\varphi(0)\|^2 \tag{3.2}$$

And using Holder's inequality and assumption **(F)**, we have

$$I_2(t)\le 3\int_0^t\| S_\alpha(t-s)\|ds\int_0^t\| S_\alpha(t-s)\|E\| f(s,u_{ns})\|_X^2\,ds$$
$$\le 3\int_0^t\| S_\alpha(t-s)\|ds\int_0^t\| S_\alpha(t-s)\|\zeta(s)E|u_{ns}|_{\mathrm{B}}^2\,ds\,.$$

Because of the fact that. $|u_t|_{\mathrm{B}}\le K(t)\sup_{r\in[0,t]}u(r)+M(t)|\varphi|_{\mathrm{B}}$. Then

$$E|u_s|_{\mathrm{B}}^2\le E\left(K(s)\sup_{r\in[0,s]}u(r)+M(s)|\varphi|_{\mathrm{B}}\right)^2$$
$$\le 2K^2(s)\sup_{r\in[0,s]}E\| u(r)\|^2+2M^2(s)E|\varphi|_{\mathrm{B}}^2.$$

So, we have

$$I_2(t)\le 6\sup_{t\ge0}\{E\| u_n(t)\|^2\}\int_0^t\| S_\alpha(t-s)\|ds\int_0^t\| S_\alpha(t-s)\|\zeta(s)K^2(s)ds$$
$$+6E|\varphi|_{\mathrm{B}}^2\int_0^t\| S_\alpha(t-s)\|ds\int_0^t\| S_\alpha(t-s)\|\zeta(s)M^2(s)ds$$
$$\le n\Delta_\infty+6E|\varphi|_{\mathrm{B}}^2\int_0^t\| S_\alpha(t-s)\|ds\int_0^t\| S_\alpha(t-s)\|\zeta(s)M^2(s)ds,$$

(3.3), where

$$\Delta_\infty=\sup_{t\ge0}\left\{6\int_0^t\| S_\alpha(t-s)\|ds\int_0^t\| S_\alpha(t-s)\|\zeta(s)K^2(s)ds\right\}$$

.

Because of Lemma 2.3.1 and assumption **(G),** we have

$$I_3(t)\le 6Ht^{2H-1}\int_0^t\| S_\alpha(t-s)\|^2\| g(s)\|_{L_2^0}^2\,ds$$
$$\le 6HC\frac{t^{2H-1}}{1+\mu(\frac{t}{2})^\alpha}\int_0^{t/2}\| S_\alpha(t-s)\|\,\| g(s)\|_{L_2^0}^2ds$$
$$+H2^{2H+2}\int_{t/2}^t\| S_\alpha(t-s)\|^2s^{2H-1}\| g(s)\|_{L_2^0}^2\,ds. \tag{3.4}$$

Combining (3.1) - (3.4) we obtain $E\| z_n(t)\|^2\le n\Delta_\infty+Q(t)$, where

$$Q(t)=3\tilde{n}^2(S_\alpha^\infty)^2E|\varphi|_{\mathrm{B}}^2$$
$$+6E|\varphi|_{\mathrm{B}}^2\int_0^t\| S_\alpha(t-s)\|ds\int_0^t\| S_\alpha(t-s)\|\zeta(s)M^2(s)ds$$
$$+6HC\frac{t^{2H-1}}{1+\mu(\frac{t}{2})^\alpha}\int_0^{t/2}\| S_\alpha(t-s)\|\,\| g(s)\|_{L_2^0}^2ds$$
$$+H2^{2H+2}\int_{t/2}^t\| S_\alpha(t-s)\|^2s^{2H-1}\| g(s)\|_{L_2^0}^2\,ds,$$

be a uniformly bounded function, thanks to assumptions (B), (G), $S_\alpha(t)$ is uniformly bounded and the fact that

$$\sup_{t\ge0}\int_0^t\| S_\alpha(t-s)\|ds=\sup_{t\ge0}\int_0^t\| S_\alpha(r)\|dr$$
$$\le\sup_{t\ge0}\int_0^t\frac{C}{1+\mu r^\alpha}dr<\infty.$$

Hence

$$1<\sup_{t\ge 0}\frac{E\parallel z_n(t)\parallel^2}{n}\le \sup_{t\ge 0}\frac{Q(t)}{n}+\Delta_\infty .$$

Passing to the limit in the last inequality as $n\to\infty$, we get a contradiction with (3.1). Second, we prove that there exists $\rho>0$ such that $\mathcal{F}(B^{\vartheta}_{R,\varphi}(\rho))\subset B^{\vartheta}_{R,\varphi}(\rho)$. Assume to the contrary that for each $n\in\mathbb{N}$ there exists $u_n\in B_R$ satisfying that $\sup_{t\ge 0} t^{\vartheta}E\parallel u_n(t)\parallel^2\le n$ but $z_n=\mathcal{F}(u_n)\notin B^{\vartheta}_{R,\varphi}(n)$, it means that $\sup_{t\ge 0} t^{\vartheta}E\parallel z_n(t)\parallel^2>n$. For $t\ge 0$, we have

$$t^{\vartheta}E\parallel z_n(t)\parallel^2\le 3t^{\vartheta}E\parallel S_\alpha(t)\varphi(0)\parallel^2$$
$$+3t^{\vartheta}E\parallel \int_0^t S_\alpha(t-s)f(s,u_{ns})ds\parallel^2$$
$$+3t^{\vartheta}E\parallel \int_0^t S_\alpha(t-s)g(s)dW^H(s)\parallel^2$$
$$=3(P_1(t)+P_2(t)+P_3(t)),$$

where

$$P_1(t)=t^{\vartheta}E\parallel S_\alpha(t)\varphi(0)\parallel^2,$$
$$P_2(t)=t^{\vartheta}E\parallel \int_0^t S_\alpha(t-s)f(s,u_{ns})ds\parallel^2;$$
$$P_3(t)=t^{\vartheta}E\parallel \int_0^t S_\alpha(t-s)g(s)dW^H(s)\parallel^2 .$$

Considering $\parallel S_\alpha(t)\parallel\le \frac{C}{1+\mu t^\alpha},\forall t\ge 0,$ then

$$P_1(t)\le \frac{t^{\vartheta}C^2E\parallel \varphi(0)\parallel^2}{(1+\mu t^\alpha)^2}.$$

(3.5)

Through assumptions (F), we have

$$P_2(t)\le t^{\vartheta}\int_0^t\parallel S_\alpha(t-s)\parallel ds\int_0^t\parallel S_\alpha(t-s)\parallel\zeta(s)E\,|u_{ns}|^2_{\mathrm{B}}\,ds$$
$$\le 2t^{\vartheta}\sup_{t\ge 0}\{E\parallel u_n(t)\parallel^2\}\int_0^t\parallel S_\alpha(t-s)\parallel ds\int_0^t\parallel S_\alpha(t-s)\parallel\zeta(s)K^2(s)ds$$
$$+2E|\varphi|^2_{\mathrm{B}}\,t^{\vartheta}\int_0^t\parallel S_\alpha(t-s)\parallel ds\int_0^t\parallel S_\alpha(t-s)\parallel\zeta(s)M^2(s)ds$$

$$\le\frac{1}{3}n\Delta_\infty+2E|\varphi|^2_{\mathrm{B}}\frac{Ct^{\vartheta}}{1+\mu\left(\frac{t}{2}\right)^\alpha}\int_0^t\parallel S_\alpha(t-s)\parallel ds\int_0^{t/2}\zeta(s)M^2(s)ds$$
$$+2^{\vartheta+1}E|\varphi|^2_{\mathrm{B}}\int_0^t\parallel S_\alpha(t-s)\parallel ds\int_{t/2}^t\parallel S_\alpha(t-s)\parallel s^{\vartheta}\zeta(s)M^2(s)ds.$$

(3.6)

In addition,

$P_3(t)\le 2Ht^\alpha\int_0^t\parallel S_\alpha(t-s)\parallel^2\parallel g(s)\parallel^2_{\mathrm{L}_2^0}\,ds,$ it's because of that $\alpha=\vartheta+(2H-1)$.

Similar to previous estimate, ones get

$$P_3(t)\quad\le 2HC\frac{t^\alpha}{1+\mu\left(\frac{t}{2}\right)^\alpha}\int_0^{t/2}\parallel S_\alpha(t-s)\parallel\parallel g(s)\parallel^2_{\mathrm{L}_2^0}ds$$
$$+2^{\alpha+1}H\int_{t/2}^t\parallel S_\alpha(t-s)\parallel^2 s^\alpha\parallel g(s)\parallel^2_{\mathrm{L}_2^0}\,ds.$$

(3.7)

Combining (3.5)- (3.7) we obtain

$$t^{\vartheta}E\parallel z_n(t)\parallel^2\le n\Delta_\infty+3G(t),$$

(3.8)

where

$$G(t)=\frac{t^{\vartheta}\tilde{n}^2C^2E|\varphi|^2_{\mathrm{B}}}{(1+\mu t^\alpha)^2}$$
$$+2E|\varphi|^2_{\mathrm{B}}\frac{Ct^{\vartheta}}{1+\mu\left(\frac{t}{2}\right)^\alpha}\int_0^t\parallel S_\alpha(t-s)\parallel ds\int_0^t\zeta(s)M^2(s)ds$$
$$+2^{\vartheta+1}E|\varphi|^2_{\mathrm{B}}\int_0^t\parallel S_\alpha(t-s)\parallel ds\int_0^t\parallel S_\alpha(t-s)\parallel s^{\vartheta}\zeta(s)M^2(s)ds$$

$$+2HC\frac{t^\alpha}{1+\mu\left(\frac{t}{2}\right)^\alpha}\int_0^{t/2}\parallel S_\alpha(t-s)\parallel\parallel g(s)\parallel^2_{\mathrm{L}_2^0}ds$$
$$+2^{\alpha+1}H\int_{t/2}^t\parallel S_\alpha(t-s)\parallel^2 s^\alpha\parallel g(s)\parallel^2_{\mathrm{L}_2^0}\,ds,$$

which is uniformly bounded. It follows from (3.8) that

$$1<\frac{t^{\vartheta}E\parallel z_n(t)\parallel^2}{n}\le\frac{1}{n}3G(t)+\Delta_\infty .$$

We get a contradiction with (3.1) by passing to the limit in the last inequality as $n\to\infty$. The proof is complete.

Theorem 3.1. If conditions in Lemma 3.1 hold, then the mild solution to (1.1) exists uniquely and decays to zero in mean square moment, i.e., $E\parallel u(t)\parallel^2\to 0,$ as $t\to\infty$.

Proof

First of all, for $u\in C\left([0,+\infty),L^2(\Omega,\ell^2)\right)$ we check that the mapping $t\to\mathcal{F}(u)(t)$ belongs to the space $C\left([0,+\infty),L^2(\Omega,\ell^2)\right)$. To do that, let $t_1,t_2>0$. We have

$$E\left\|\int_0^{t_1}S_\alpha(t_1-s)g(s)dW^H(s)-\int_0^{t_2}S_\alpha(t_2-s)g(s)dW^H(s)\right\|^2$$
$$\le 2E\left\|\int_0^{t_1}(S_\alpha(t_1-s)-S_\alpha(t_2-s))g(s)dW^H(s)\right\|^2$$
$$+2E\left\|\int_{t_1}^{t_2}S_\alpha(t_2-s)g(s)dW^H(s)\right\|^2:=J_1+J_2.$$

Applying Lemma 2.3.1 to J_1,

$$J_1 \le 4Ht_1^{2H-1}\int_0^{t_1}\left\|\left[S_\alpha(t_1-s)-S_\alpha(t_2-s)\right]g(s)\right\|_{L_2^0}^2 ds$$
$$\le 4Ht_1^{2H-1}\int_0^{t_1}\left\|S_\alpha(t_1-s)-S_\alpha(t_2-s)\right\|^2 \| g(s)\|_{L_2^0}^2 ds.$$

It can be considered that t_1 is fixed and $t_2 = t_1 + h$. Denote

$$Q_h(s) = \left\|S_\alpha(t_1-s)-S_\alpha(t_2-s)\right\|^2 \| g(s)\|_{L_2^0}^2,$$

then $Q_h(s) \to 0$ as $h \to 0$ for all $s \in (0,t_1)$.

In addition

$$Q_h(s) \le 2(\| S_\alpha(t_1-s)\|^2 + \| S_\alpha(t_2-s)\|^2)\| g(s)\|_{L_2^0}^2$$
$$\le 4(S_\alpha^\infty)^2 \| g(s)\|_{L_2^0}^2 := Q(s) \in L^1(0,t_1),$$

we conclude, by Lebesgue dominated convergence theorem that,

$$\int_0^{t_1} Q_h(s)ds \to 0 \text{ as } h \to 0.$$

It leads to $J_1 \to 0$ when $|t_2 - t_1| \to 0$. Now applying Lemma 2.3.1 to J_2 we obtain

$$J_2 \le 4H|t_2-t_1|^{2H-1}\int_{t_1}^{t_2}\left\|S_\alpha(t_2-s)g(s)\right\|_{L_2^0}^2 ds$$
$$\le 4(S_\alpha^\infty)^2 H|t_2-t_1|^{2H-1}\int_{t_1}^{t_2}\left\|g(s)\right\|_{L_2^0}^2 ds \to 0,$$

when $|t_2 - t_1| \to 0$. The proof is therefore complete, thanks to the continuity of $t \to S_\alpha(t)\varphi(0)$ and $t \to \int_0^t S_\alpha(t) f(s,u_s)ds$.

To get desired results, it is enough to show that the operator $\mathcal{F}$ has a unique fixed point in $B_{R,\varphi}^{\vartheta}(\rho)$. For $u, v \in \mathbf{BC}_\varphi$ and the formula of $\mathcal{F}$, we have

$$E\| \mathcal{F}(u)(t) - \mathcal{F}(v)(t)\|^2$$
$$\le E\left\| \int_0^t S_\alpha(t-s)\left(f(s,u_s)-f(s,v_s)\right)ds\right\|^2.$$

Noting that

$$E|u_s - v_s|_{\mathcal{B}}^2 \le E\left\| K(s)\sup_{r\in[0,s]}\left(u(r)-v(r)\right)\right\|^2$$
$$\le K^2(s)\sup_{r\in[0,s]} E\| u(r)-v(r)\|^2,$$

thanks to assumption (B). Then

$$E\| \mathcal{F}(u)(t) - \mathcal{F}(v)(t)\|^2 = E\left\| \int_0^t S_\alpha(t-s)\left(f(s,u_s)-f(s,v_s)\right)ds\right\|^2$$
$$\le \int_0^t \| S_\alpha(t-s)\| ds \int_0^t \| S_\alpha(t-s)\| \zeta(s) E|u_s - v_s|_{\mathcal{B}}^2 ds$$
$$\le \int_0^t \| S_\alpha(t-s)\| ds \int_0^t \| S_\alpha(t-s)\| \zeta(s)K^2(s)\sup_{r\in[0,s]} E\| u(r)-v(r)\|^2 ds$$
$$\le \left(\int_0^t \| S_\alpha(t-s)\| ds \int_0^t \| S_\alpha(t-s)\| \zeta(s)K^2(s)ds\right)\sup_{t\ge 0} E\| u(t)-v(t)\|^2$$
$$\le \frac{1}{6}\Delta_\infty \sup_{t\ge 0} E\| u(t)-v(t)\|^2. \qquad (3.9)$$

By the estimate (3.9), we obtain that

$$\| \mathcal{F}(u) - \mathcal{F}(v)\|_{BC}^2 \le \frac{1}{6}\Delta_\infty \| u - v\|_{BC}^2.$$

The last relation ensures that the solution operator $\mathcal{F}$ is a contraction mapping on $\mathbf{BC}_\varphi$. By Lemma 3.1 we have the conclusion due to Banach fixed point principle.

Remark 3.1. The assumptions (A), (F), (G) are very popular in the context of solvability problems. Consider the phase space $\mathcal{B} = C_\gamma$ with γ is a positive number, where the norm in $\mathcal{B}$ is given by $|w|_{\mathcal{B}} = \sup_{\theta\le 0} e^{\gamma\theta}\| \varphi(\theta)\|$. Thanks to the formula (2.3), we have

$$K(t) = 1, M(t) = e^{-\gamma t}.$$

So one observes that K and M are uniformly bounded, $\int_0^\infty s^{\vartheta} M^2(s) < \infty$ and then assumption {\bf (B)} is satisfied with $M^2(t) = o(1)$ as $t \to \infty$. Condition (3.1) will be satisfied with a suitable function $\zeta \in L_{loc}^1(\mathbb{R}^+)$. For example, we can choses $\zeta(t) = \frac{\epsilon}{1+t}$ with $\epsilon > 0$ is small enough.

4. CONCLUSION

In this paper, we prove that the mild solution to (1.1) exists uniquely and decays to zero in mean square moment by applying the fixed points method. After that, we give a remark to the abstract results.

REFERENCE

[1] E. Alos, O. Mazet, and D. Nualart, Stochastic calculus with respect to Gaussian processes, *Annals of Probability* vol. 29, no. 2, pp. 766–801, 2001.

[2] B. Boufoussi, S. Hajji, Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space, *Statistics and Probability Letters* 82 (2012) 1549-1558.

[3] T. Caraballo, M.J. Garrido-Atienza, T. Taniguchi, The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion, *Nonlinear Analysis* 74 (2011) 3671-3684.

[4] E. Cuesta, Asymptotic behaviour of the solutions of fractional integro-differential equations and some time discretizations, *Discrete Contin. Dyn. Syst.* (Supplement) (2007) 277-285.

[5] J. Hale, J. Kato, Phase space for retarded equations with infinite dealy, *Funkcial. Ekvac.* 21 (1978),11-41.

[6] J.K. Hale, Numerical dynamics, Chaotic Numerics, Contemporary Mathematics, vol. 172, *American Mathematical Society*, Providence, RI, 1994, pp. 1–30.

[7] Y. Hino, S. Murukami, T. Naito, Functional Differential Equations with Unbounded Delay, *Lecture Notes in Mathematics*, vol. 1473, Springer-Verlag, Berlin, 1991.

[8] D. Nualart, The Malliavin Calculus and Related Topics, Second Edition, *Springer-Verlag*, Berlin, 2006.

NGHIỆM PHÂN RÃ CỦA PHƯƠNG TRÌNH VI TÍCH PHÂN NGẪU NHIÊN VỚI CHUYỂN ĐỘNG BROWN BẬC PHÂN THỨ

Nguyễn Như Quân[1], Nguyễn Ngọc Viên[2]

[1]*Khoa Khoa học tự nhiên, Đại học Điện lực*

[2]*Khoa Cơ bản, Đại học Hải Dương*

Tóm tắt: *Trong công trình này, chúng tôi chứng minh sự tồn tại nghiệm phân rã theo nghĩa bình phương trung bình của một lớp phương trình vi tích phân ngẫu nhiên với trễ vô hạn và chuyển động Brown bậc phân số. Sự tồn tại nghiệm nhẹ đạt được bằng việc sử dụng Định lí điểm bất động Banach và các kỹ thuật ước lượng bất đẳng thức thích hợp.*

Từ khóa: *Chuyển động Brown bậc phân số; Phương trình vi tích phân; Định lí điểm bất động; Trễ vô hạn.*

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Liên lạc: Nguyễn Như Quân, e - mail: quan2n@epu.edu.vn