A NOTE ON THE ENDOMORPHISM RING OF ORTHOGONAL MODULES

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Received on 25/4/2019, accepted for publication on 13/6/2019

Abstract: In this paper, we extend Mohamed−Müller's results [2, Lemma 3.3] about the endomorphism ring of a module $M = \oplus_{i\in I} M_i$, where M_i and M_j are orthogonal for all distinct elements $i, j \in I$.

1 Introduction

All rings are associated with identity, and all modules are unital right modules. The endomorphism ring of M are denoted $End(M)$. A submodule N of M is said to be an *essential* (notationally $N \subset^e M$) if $N \cap K \neq 0$ for every nonzero submodule K of M. Two modules M and N are called *orthogonal* if they have no nonzero isomorphic submodules. Let N be a right $R-$module. A module M is said to be $N-$*injective* if for every submodule X of N, any homomorphism $\varphi : X \longrightarrow M$ can be extended to a homomorphism $\psi : N \longrightarrow M$. Two modules M and N are called *relatively injective* if M is $N-$injective and N is $M-$injective. In [2, Lemma 3.3], S. H. Mohamed and B. J. Müller proved that:

Let $M = M_1 \oplus M_2$. If M_1 and M_2 are orthogonal, then

$$S/\Delta \cong S_1/\Delta_1 \times S_2/\Delta_2.$$

The converse holds if M_1 and M_2 are relatively injective, where

$$S = End(M), S_i = End(M_i)(i = 1, 2)$$

and

$$\Delta = \{s \in S \mid Ker(s) \subset^e M\}, \Delta_i = \{s_i \in S_i \mid Ker(s_i) \subset^e M_i\}(i = 1, 2).$$

In this paper, we study [2, Lemma 3.3] in generalized case. We have:

Theorem A. *(i). Let $M = \oplus_{i\in I} M_i$ be a direct sum of submodules such that M_i and M_j are orthogonal for any i, j of I and $i \neq j$, then $\prod_{i\in I} S_i/\Delta_i$ is embedded into S/Δ.*

In particular, if I is a finite set, $\prod_{i\in I} S_i/\Delta_i \cong S/\Delta$.

(ii). Let $M = \oplus_{i\in I} M_i$ be a direct sum of submodules such that M_i and M_j are relatively injective for any i, j of I, $i \neq j$ and $\prod_{i\in I} S_i/\Delta_i \cong S/\Delta$, then M_i and M_j are orthogonal with i, j of I and $i \neq j$, where $S = End(M)$, $S_i = End(M_i)(i \in I)$ and $\Delta = \{s \in S \mid Ker(s) \subset^e M\}$, $\Delta_i = \{s_i \in S_i \mid Ker(s_i) \subset^e M_i\}(i \in I)$.

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2 Proof of Theorem A

(i). Let s be an element of the endomorphism ring S and x an element of the module M, then $x = \sum_{i\in I} x_i$ with $x_i \neq 0$ for every $i \in I'$ (where I' is the finite subset of I), $s(x) = \sum_{i\in I} s(x_i)$. Because $s(x_i)$ is an element of M, thus $s(x_i) = \sum_{j\in I} s_{ij}(x_i)$ with $s_{ij}(x_i) = p_j \circ s(x_i)$ is an element of M_j (where $p_j : M \longrightarrow M_j$ is a natural homomorphism, $s_{ij}(x_i) \neq 0$ for every $j \in I_0$, I_0 is finite and I_0 is a subset of I). We consider the matrix $s = [s_{ij}]_{I\times I}$ with $s_{ij} : M_i \longrightarrow M_j$ being homomorphism. Note that, s_{ij} is an endomorphism of M because $s_{ij}(\sum_{j\in I} x_j) = 0 + ... + 0 + s_{ij}(x_i) + 0 + ...$

Claim 1. $Ker(s_{ij})$ is an essential submodule of M for every i, j belonging to I and $i \neq j$.

Let N be a nonzero submodule of M and $Ker(s_{ij}) \cap N = 0$, then $s_{ij} |_N$ is a monomorphism, thus $N \cong s_{ij}(N)$ with $s_{ij}(N)$ being a submodule of M_j. But $s_{ij}(\oplus_{k\neq i} M_k) = 0$, thus $\oplus_{k\neq i} M_k$ is a submodule of $Ker(s_{ij})$. Hence $\oplus_{k\neq i} M_k \cap N = 0$, $(\oplus_{k\neq i} M_k) \oplus N$ is a submodule of $M = (\oplus_{k\neq i} M_k) \oplus M_i$. Thus

$$N \cong ((\oplus_{k\neq i} M_k) \oplus N)/(\oplus_{k\neq i} M_k) \subset M/(\oplus_{k\neq i} M_k) \cong M_i.$$

Let $s_{ij}(N) = Y$ be a submodule of M_j, there exists a submodule X of M_i such that $X \cong N \cong Y$. This is a contradiction to the fact that M_i and M_j are orthogonal. Therefore, $Ker(s_{ij})$ is an essential submodule of M for every i, j that are elements of I and $i \neq j$.

Claim 2.

$$Ker(s) \cap M_i = \cap_{j\in I} Ker(s_{ij}),$$

for every i of I.

Let $s : \oplus_{i\in I} M_i \longrightarrow \oplus_{i\in I} M_i$, and let x be an element of $\oplus_{i\in I} M_i$, then $x = \sum_{i\in I} x_i$ with $x_i \in M_i$, $x_i \neq 0$ for every $i \in I'$ (where I' is finite and I' is subset of I). Thus

$$s(x) = s(\sum_{i\in I} x_i) = \sum_{i\in I} s(x_i) = \sum_{i\in I} \sum_{j\in I} s_{ij}(x_i) = [s_{ij}]^T_{I\times I}.[x_i]_{I\times 1},$$

with $[s_{ij}]^T_{I\times I}$ is the transposet matrix of $[x_{ij}]_{I\times I}$. Let x be an element of $Ker(s) \cap M_i$, then x is an element of M_i and $s(x) = 0$. Thus $x = \sum_{j\in I} x_j = x_i$ with x_j being an element of M_j for every j of I, and $s_{ij}(x_i) = 0$ for every j of I. Hence, x_i is an element of $Ker(s_{ij})$ for every I, it follows that x is an element of $\cap_{j\in I} Ker(s_{ij})$, thus $Ker(s) \cap M_i$ is a subset of $\cap_{j\in I} Ker(s_{ij})$. If x is an element of $\cap_{j\in I} Ker(s_{ij})$ then x is an element of M_i and $s_{ij}(x) = 0$ for every j in I. Thus $s(x) = s(\sum_{j\in I} x_j) = s(x_i) = \sum_{j\in I} s_{ij}(x_i) = 0$, hence x is an element of $Kers$, i.e., x is an element of $Ker(s) \cap M_i$. It follows that $\cap_{j\in I} Ker(s_{ij})$ is a subset of $Ker(s) \cap M_i$. Thus,

$$Ker(s) \cap M_i = \cap_{j\in I} Ker(s_{ij}),$$

for every i of I.

Claim 3. If s is an element of Δ then s_i is an element of Δ_i, for every i of I.

Let s be an element of Δ, then $Ker(s)$ is an essential submodule of M. By Claim 2 and [1, Proposition 5.16], $Ker(s) \cap M_i = \cap_{j\in I} Ker(s_{ij})$ is an essential submodule of M_i for every i of I. Thus $Ker(s_i)$ is an essential submodule of M_i. It follows that s_i is an element of Δ_i, for every i of I.

Claim 4. If I is a finite set and s_i is an element of Δ_i for every i of I then s is also an element of Δ.

By Claim 1, $Ker_{i\neq j}(s_{ij})$ is an essential submodule of M for every i of I, thus $Ker_{i\neq j}(s_{ij})\cap M_i$ is also an essential submodule of M_i. Since I is the finite set and by [1, Proposition 5.16], $\cap_{i\neq j}Ker(s_{ij})$ is an essential submodule of M_i. Because, s_i is an element of Δ_i, $Ker(s_i)$ is an essential submodule of M_i, thus $\cap_{j\in I}Ker(s_{ij})$ is an essential submodule of M_i for every i of I. Hence $Ker(s)\cap M_i$ is an essential submodule of M_i (by Claim 2), $\oplus_{i\in I}(Ker(s)\cap M_i)$ is an essential submodule of $M=\oplus_{i\in I}M_i$. Thus $Ker(s)$ is also an essential submodule of M. It follows that s is an element of Δ.

By Claim 1, Claim 2, Claim 3,

$$S/\Delta = (A_{ij})_{I\times I}$$

with $A_{ij} = S_i/\Delta_i$ if $i=j$ and $A_{ij}=0$ if $i\neq j$. Let $\varphi : \prod_{i\in I} S_i/\Delta_i \longrightarrow S/\Delta$ be a homomorphism such that $\varphi((s_i+\Delta_i)) = [s_{ij}]_{I\times I}$ with s_{ij} is an element of A_{ij}. Note that $Ker(\varphi) = \{(s_i+\Delta_i) \mid s=[s_{ij}]_{I\times I}\in\Delta\} = \{(s_i+\Delta_i)\mid s_i\in\Delta_i\} = (0)$, thus φ is a monomorphism. Hence, $\prod_{i\in I} S_i/\Delta_i \cong X$ with X is a submodule of S/Δ.

If I is a finite set, then s is an element of Δ if and only if s_i is an element of Δ_i for every i of I. Hence $S/\Delta = [A_{ij}]_{I\times I}\cong \prod_{i\in I} S_i/\Delta_i$.

(ii). Assume that, $\prod_{i\in I} S_i/\Delta_i \cong S/\Delta$ with M_i and M_j are relatively injective for every i,j are elements of I and $i\neq j$, we will show that M_i and M_j are orthogonal for any i,j of I and $i\neq j$.

Assume that, there are two elements α and β of I and $\alpha\neq\beta$ such that M_α and M_β are not orthogonal. There exist two submodules E_α of M_α and E_β of M_β with $E_\alpha\cong E_\beta$. Let $f_{\alpha\beta}: E_\alpha\longrightarrow E_\beta$ be an isomorphism, then $f_{\alpha\beta}: E_\alpha\longrightarrow M_\beta$ is a monomorphism. Since M_β is $M_\alpha-$injective, there exist $g_{\alpha\beta}: M_\alpha\longrightarrow M_\beta$ is an extending of $f_{\alpha\beta}$. Note that $Ker(g_{\alpha\beta})$ is an essential submodule of M thus $Ker(g_{\alpha\beta})\cap E_\alpha\neq 0$. There exists element x_α of E_α with $x_\alpha\neq 0$ and $g_{\alpha\beta}(x_\alpha) = f_{\alpha\beta}(x_\alpha)=0$, this is the contradiction. Since f is a monomorphism. Hence, M_i and M_j are orthogonal for any i,j of I and $i\neq j$.

By the Theorem A, we have the Corollary B.

Corollary B. *(i). Let $M=\oplus_{i=1}^n M_i$ be a direct sum of submodules such that M_i and M_j are orthogonal for any i,j of $\{1,2,...,n\}$ and $i\neq j$, then $\prod_{i=1}^n S_i/\Delta_i\cong S/\Delta$.*

(ii). Let $M=\oplus_{i=1}^n M_i$ be a direct sum of submodules such that M_i and M_j are relatively injective for any i,j of $\{1,2,...,n\}$, $i\neq j$ and $\prod_{i=1}^n S_i/\Delta_i\cong S/\Delta$, then M_i and M_j are orthogonal with i,j of $\{1,2,...,n\}$ and $i\neq j$, where $S=End(M)$, $S_i=End(M_i)(i=1,2,...,n)$ and $\Delta=\{s\in S\mid Ker(s)\subset^e M\}$, $\Delta_i=\{s_i\in S_i\mid Ker(s_i)\subset^e M_i\}(i=1,2,...,n)$.

Note that, Regarding Corollary B, in case $n=2$, we have [2, Lemma 3.3].

Acknowledgment

This research was supported by Ministry of Education and Training, grant no. B2018-HHT-02.

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TÓM TẮT

MỘT CHÚ Ý VỀ VÀNH CÁC TỰ ĐỒNG CẤU CỦA MÔĐUN TRỰC GIAO

Trong bài báo này chúng tôi đưa ra một kết quả về vành các tự đồng cấu của môđun $M = \oplus_{i\in I} M_i$ trong đó M_i và M_j là trực giao lẫn nhau với bất kỳ i, j của I và $i \neq j$. Kết quả này đã tổng quát một kết quả của S. H. Mohamed và B. J.Müller trong [2, Lemma 3.3].