VISCOSITY SOLUTIONS OF THE AUGMENTED $K-$ HESSIAN EQUATIONS

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Abstract: *In this paper, the Dirichlet problem for the augmented $k-$ Hessian equations in the bounded domain with nonsmooth data will be investigated. We introduce the concept of $(\omega,k)-$ convex function, show that all viscosity subsolutions and supersolutions of the considering Dirichlet problem are $(\omega,k)-$ convex. Furthermore, we prove some sufficient conditions for the existence and uniqueness of the viscosity solutions of the Dirichlet problem.*

Keywords: *augmented $k-$ Hessian equations, viscosity solutions.*

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\mathbb{M}^n$ be the set of all $n\times n$ symmetric matrices with the norm given by $\| X \| = \max |x_{ij}|$; for $X,Y \in \mathbb{M}^n, X \le Y$ means that $\lambda_i \le \mu_i, i=1,2,\ldots,n$, where $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ and $\mu_1 \le \mu_2 \le \ldots \le \mu_n$ are eigenvalues of $X,Y,$ respectively; I is the unit matrix of order $n.$ In this paper, we study the Dirichlet problem for the augmented $k-$ Hessian equations

($k \in \{1,2,\ldots,n\}$) of the form:

$$-[\sigma_k(\mu(D^2 v-\omega(x,v,Dv)))]^{1/k} + f(x,v,Dv)=0, \quad x\in\Omega, \tag{1}$$

$$v(x)=\psi(x), \quad x\in\partial\Omega, \tag{2}$$

where $\omega:\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{M}^n$ and $f:\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ are given continuous functions,

$f>0;$ $\mu(X)=(\mu_1,\cdots,\mu_n)$ are n eigenvalues of $X\in\mathbb{M}^n$;

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$$\sigma_k(\mu_1,\cdots,\mu_n)=\sum_{1\le i_1<\cdots<i_k\le n}\mu_{i_1}\cdots\mu_{i_k}$$

are basic symmetric polynomials of degree k; ψ is a given continuous function defined on $\partial\Omega$. The minus sign in (1) was chosen to get the degenerate ellipticity of the equation.

If $k=n$ and $\omega=0,$ Equation (1) becomes a Monge-Ampere equation

$$-\det D^2v+[f(x,v,Dv)]^n=0,\quad x\in\Omega.$$

If $k=1$ and $\omega=0,$ Equation (1) becomes a nonlinear Poisson equation

$$-\Delta v+f(x,v,Dv)=0,\quad x\in\Omega.$$

Monge-Ampere equations, and Poisson equations in particular, $k-$ Hessian equations in general have many applications in various fields including Physics, Geometric Curvatures, etc. [2], [5]-[8].

If the data of the problem are sufficiently smooth, classical solutions of Dirichlet problems for Monge-Ampere equations have been studied, even for a more general class of equations in [7], [8]. Meanwhile, classical solutions to (1)-(2) have been investigated in [5] and further extended for oblique boundary value problems for the augmented Hessian equations in [6]. If the data of the problem are nonsmooth, we need to study its generalized solutions. The viscosity solutions for (1)-(2) have been studied by A. Colesanti [2] in the case that $\omega=0$ and f depends on x only. In this paper, we extend several results of A. Colesanti for the general case mentioned above.

We first recall the notions and some essential results on viscosity solutions of elliptic second order partial differential equations in finite dimensional space. A complete theory can be found in [1]. To be more specific, consider the Dirichlet problem

$$F(x,v,Dv,D^2v)=0,\ \text{in }\Omega;\quad v=\psi\ \text{on }\partial\Omega, \tag{3}$$

where ψ is a continuous function on $\partial\Omega$; F is a real-valued continuous function on $\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{M}^n$, and satisfies the following two conditions

$$F(x,t,p,X)\le F(x,t,p,Y),\quad \forall X\ge Y \tag{4}$$

(this condition is also known as the *degenerate ellipticity* of F) and

$$F(x,t,p,X)\ge F(x,s,p,X),\quad \forall(x,p,X)\in\Omega\times\mathbb{R}^n\times\mathbb{M}^n,\ \forall t>s. \tag{5}$$

A sufficient condition for (5) is: for each $0 < R < \infty$, there exists a constant $C_R > 0$:

$$F(x,t,p,X) \geq F(x,s,p,X) + C_R(t-s), \tag{6}$$

for all $x \in \Omega, R \geq t \geq s \geq -R, p \in \mathbb{R}^n, X \in \mathbb{M}^n$.

Regarding the dependence on x, we need the following assumption: for each $0 < R < \infty$, there is a real-valued continuous and nondecreasing function $\gamma_R(\tau)$ satisfying $\gamma_R(\tau) \to 0$ as $\tau \to 0^+$ such that

$$|F(x,t,p,X) - F(y,t,p,X)| \leq \gamma_R(|x-y|(1+|p|)), \tag{7}$$

for any $x, y \in \Omega, |t| \leq R, p \in \mathbb{R}^n, X \in \mathbb{M}^n$.

We say that the function φ touches the function v from above (resp. below) at $x_0 \in \Omega$, if $v - \varphi$ attains its local maximum (resp. local minimum) at x_0 and $v(x_0) = \varphi(x_0)$.

The definition of viscosity solutions of (3) is given below.

Definition 1.1 ([1]). a) An upper semi-continuous function v on $\bar{\Omega}$ is said to be a *viscosity subsolution* of the equation in (3) if for any $\varphi \in C^2(\Omega)$ touching v from above at $x_0 \in \Omega$, we have

$$F(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0.$$

b) A lower semi-continuous function on $\bar{\Omega}$ is said to be a *viscosity supersolution* of the equation in (3) if for any $\varphi \in C^2(\Omega)$ touching v from below at $x_0 \in \Omega$ we have

$$F(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0.$$

c) A function v is a *viscosity solution* of the equation in (3) if v is both a viscosity subsolution and a viscosity supersolution of it.

The existence and uniqueness of the viscosity solution of (3) has been established by H. Ishii in the following result.

Theorem 1.2 ([4], Theorem II.1, Proposition II.1). *Let F satisfy the conditions (4), (6) and (7). If the equation in (3) has a viscosity subsolution v_1 and a viscosity supersolution v_2 being locally Lipschitz on Ω and $v_1 = v_2 = \psi$ on $\partial\Omega$ then, there exists a unique viscosity solution of the problem (3).*

Based on the preceding theorem, we derive several sufficient conditions for the existence of a unique viscosity solution of the problem (1)-(2).

2. RESEARCH CONTENT

Let $H_k(\mu_1,\cdots,\mu_n)=[\sigma_k(\mu_1,\cdots,\mu_n)]^{1/k}$ and

$$F_k(x,v,Dv,D^2v)=-H_k(\mu(D^2v-\omega(x,v,Dv)))+f(x,v,Dv).$$

Then Equation (1) becomes

$$F_k(x,v,Dv,D^2v)=0,\quad x\in\Omega.$$

Let

$$\Gamma_k:=\{\mu\in\mathbb{R}^n:\sigma_j(\mu)>0,\forall j=1,2,\cdots,k\}.$$

It is well-known that (see [2])

$$\Gamma_n=\{\mu\in\mathbb{R}^n:\mu_j>0,\forall j=1,2,\cdots,n\},\quad \Gamma_i\subset\Gamma_j,\forall i>j.$$

Moreover, the $k-$ Hessian operator $H_k(\mu(D^2v))$ is degenerate elliptic on $\overline{\Gamma}_k$. Hence, in order to get the degenerate ellipticity of the function F_k, we need to consider the test functions $\varphi\in C^2(\Omega)$ such that $\mu(D^2\varphi(x)-\omega(x,\varphi(x),D\varphi(x)))\in\overline{\Gamma}_k$. This leads to the definition of $(\omega,k)-$convexity as below.

Definition 2.1 Given a pair (ω,k). A function $v\in C(\Omega)$ is said to be $(\omega,k)-$*convex* on Ω iff for any $\varphi\in C^2(\Omega)$, φ touches v from below at $x_0\in\Omega$ we have

$$\mu(D^2\varphi(x_0)-\omega(x_0,\varphi(x_0),D\varphi(x_0)))\in\overline{\Gamma}_k.$$

It is clear that if $v\in C^2(\Omega)$ and v is $(\omega,k)-$convex on Ω then,

$$\mu(D^2v(x)-\omega(x,v(x),Dv(x)))\in\overline{\Gamma}_k,\quad \forall x\in\Omega,$$

and for C^2-functions, the $(0,n)-$convexity is exactly the usual convexity.

The following theorem establishes the $(\omega,k)-$ convexity of the viscosity supersolutions and viscosity subsolutions of (1).

Theorem 2.2 *Suppose ω,f are given continuous functions, $f>0$. If v is a viscosity subsolution or a viscosity supersolution of (1) then, v is a $(\omega,k)-$convex function on Ω.*

Proof. We consider the case that v is a viscosity subsolution. The other case can be handled analogously. Indeed, suppose that v is a viscosity subsolution of (1) but v is not $(\omega,k)-$ convex. Then, there exist $x_0 \in \Omega$ and a function $\varphi_0 \in C^2(\Omega)$, φ_0 touches v from below at $x_0 \in \Omega$ but

$$\mu(D^2\varphi_0(x_0) - \omega(x_0, \varphi_0(x_0), D\varphi_0(x_0))) \notin \overline{\Gamma}_k.$$

Let $\varphi_\alpha(x) = \varphi_0(x) + \frac{\alpha}{2}|x - x_0|^2, \alpha > 0.$ It can be seen that $\varphi_\alpha \in C^2(\Omega),$ $\varphi_\alpha(x_0) = \varphi_0(x_0),$ $D\varphi_\alpha(x_0) = D\varphi_0(x_0),$ $D^2\varphi_\alpha(x_0) = D^2\varphi_0(x_0) + \alpha I,$ φ_α touches v from below at x_0 for any $\alpha > 0.$ Thus, by the definition of viscosity subsolution, we have

$$\sigma_k(\mu(D^2\varphi_0(x_0) + \alpha I - \omega(x_0, \varphi_0(x_0), D\varphi_0(x_0)))) \geq f(x_0, \varphi_0(x_0), D\varphi_0(x_0)) > 0. \tag{8}$$

On the other hand, for a sufficiently large α,

$$\mu(D^2\varphi_0(x_0) + \alpha I - \omega(x_0, \varphi_0(x_0), D\varphi_0(x_0))) \in \Gamma_n \subset \Gamma_k,$$

thus, we must have α_0 so that $\mu(D^2\varphi_0(x_0) + \alpha_0 I - \omega(x_0, \varphi_0(x_0), D\varphi_0(x_0))) \in \partial\Gamma_k.$

In other words, (by the continuity of σ_k), we have

$$\sigma_k(\mu(D^2\varphi_0(x_0) + \alpha_0 I - \omega(x_0, \varphi_0(x_0), D\varphi_0(x_0)))) = 0,$$

which contradicts to (8). The conclusion follows.

In view of the preceding theorem, we can assume that the test functions in the definition 1.1 are C^2-functions and they are $(\omega,k)-$convex on Ω and in the rest of this paper, we consider only the elements (x,t,p,X) satisfying $\mu(X - \omega(x,t,p)) \in \overline{\Gamma}_k.$

We proceed to provide additional conditions for ω and f to ensure the existence and uniqueness of the viscosity solution to the problem under consideration.

Matrix-valued function $\omega(x,t,p)$ satisfies the following conditions: For all $R > 0,$ there exists a continuous function, nondecreasing $\gamma_{\omega,R}$ on $[0,\infty)$ satisfies

$$\omega(x,t,p) - \omega(y,t,p) \leq \gamma_{\omega,R}(|x-y|(1+|p|))I, \tag{9}$$

for any $x, y \in \Omega,$ $|t| \leq R, p \in \mathbb{R}^n;$

$$\det(-\omega(x,t,p)) \geq [f(x,t,p)]^k, \quad (x,t,p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n; \tag{10}$$

$\omega(x,t,p)$ is increasing with respect to the variable t; that is

$$\omega(x,t,p) \geq \omega(x,s,p), \forall (x,t,p),(x,s,p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n, t \geq s. \tag{11}$$

For the function f, we assume that there exists a positive constant $C_{f,R}$ and there exists a real-valued single variable nondecreasing function $\gamma_{f,R}$, which is right continuous at 0 such that

$$f(x,t,p) \geq f(x,s,p) + C_{f,R}(t-s), \quad \forall x \in \Omega, R \geq t \geq s \geq -R, p \in \mathbb{R}^n; \tag{12}$$

$$| f(x,t,p) - f(y,t,p) | \leq \gamma_{f,R}(| x-y | (1+| p |)), \tag{13}$$

for any $x, y \in \Omega, | t | \leq R, p \in \mathbb{R}^n$.

It can be seen that (9)-(13) are satisfied if $\omega = -\alpha(x)I, f = f(x)$, $\alpha(x), f(x)$ are real-valued, Lipschitz continuous, and positive on $\bar{\Omega}$, $\alpha(x) \geq f(x)+1$.

We now establish a result on the existence and uniqueness of the viscosity solution of Dirichlet problem (1)-(2).

Theorem 2.3. *Let $f > 0$ be a continuous function. Suppose the conditions (9), (11), (12), (13) are satisfied. If the equation (1) has a viscosity subsolution v_1 and a viscosity supersolution v_2 and they are locally Lipschitz on Ω, $v_1 = v_2$ on $\partial\Omega$, then there exists a unique $(\omega,k)-$convex viscosity solution of the problem (1), (2).*

Proof. Note that the degenerate ellipticity of F_k has been established above. To complete the proof, it is sufficient to check that (6), (7) are satisfied, then the conclusion follows from Theorem 1.2.

Indeed, it follows from (11), the degenerate ellipticity of H_k and (12) that

$$\begin{aligned} F_k(x,t,p,X) - F_k(x,s,p,X) &= H_k(\mu(X-\omega(x,s,p))) - H_k(\mu(X-\omega(x,t,p))) \\ &\quad + f(x,t,p) - f(x,s,p) \\ &\geq f(x,t,p) - f(x,s,p) \\ &\geq C_{f,R}(t-s). \end{aligned}$$

Moreover, for any $0 < R < \infty$, $x, y \in \Omega, | t | \leq R, p \in \mathbb{R}^n, X \in \mathbb{M}^n$, it follows from (13) that

$$
\begin{aligned}
F_k(x,t,p,X)-F_k(y,t,p,X) &= -H_k(\mu(X-\omega(x,t,p))+f(x,t,p)\\
&\quad +H_k(\mu(X-\omega(y,t,p))-f(y,t,p)\\
&\le C_n^k\gamma_{\omega,R}(|x-y|(1+|p|))+\gamma_{f,R}(|x-y|(1+|p|))\\
&=(C_n^k\gamma_{\omega,R}+\gamma_{f,R})(|x-y|(1+|p|)).
\end{aligned}
$$

In the above estimates, we used the convexity, the homogeneity of H_k, (9), and

$X-\omega(x,t,p)\le X-\omega(y,t,p)+\gamma_{\omega,R}(|x-y|(1+|p|))I$

that

$$
\begin{aligned}
H_k(\mu(X-\omega(x,t,p))&\le H_k(\mu(X-\omega(y,t,p)+\gamma_{\omega,R}(|x-y|(1+|p|))I)\\
&\le H_k(\mu(X-\omega(y,t,p))+H_k(\gamma_{\omega,R}(|x-y|(1+|p|))I)\\
&=H_k(\mu(X-\omega(y,t,p))+C_n^k\gamma_{\omega,R}(|x-y|(1+|p|)),
\end{aligned}
$$

where C_n^k is the number of k combinations of n elements.

Interchange the role of x and y, let $\gamma_R=C_n^k\gamma_{\omega,R}+\gamma_{f,R}$ we obtain

$|F_k(x,t,p,X)-F_k(y,t,p,X)|\le\gamma_R(|x-y|(1+|p|)).$

The conclusion follows.

By Theorem 2.3, the existence and uniqueness of the viscosity solution of the problem (1)-(2) is reduced to the existence of a viscosity subsolution and a viscosity supersolution in the class of local Lipschitz continuous functions of the given problem. The following theorem provides a sufficient condition for the existence of such viscosity sub- and supersolutions.

Theorem 2.4. *Let* Ω *be a strictly convex domain with* $\partial\Omega\in C^{2,\alpha}$ *for* $0<\alpha<1; f>0$ *be a continuous function;* tr$X$ *be the trace of* $X\in\mathbb{M}^n;\omega=\omega(x,t,p)=[\omega_{ij}(x,t,p)]$ *be a matrix-valued function which is continuously differentiable, and satisfies the following conditions:*

i) $\omega_{ii}(x,z,p)=O(|p|^2)$;

ii) $\sum_{j=1}^{n}p_jD_{p_j}\omega_{ii}(x,z,p)\le O(|p|^2)$;

iii) $(D_z+|p|^{-2}\sum_{j=1}^{n}p_jD_{x_j})\omega_{ii}(x,z,p)\le o(|p|^2)$;

as $|p| \to \infty$ *uniformly for* $x \in \Omega$ *and bounded* z, *for each* $i = 1, 2, \ldots, n$;

iv) there exist $C_1, C_2 \geq 0$ *such that*

$$-\mathrm{tr}\omega(x,z,p)\mathrm{sign}z \leq C_1 |p| + C_2, \quad \forall (x,z,p).$$

Moreover, suppose that (9)-(13) are satisfied and $\psi \in C^{2,\alpha}(\partial\Omega)$. *Then there exists a unique viscosity solution of the problem (1)-(2).*

Proof. By Theorem 2.3, it is sufficient to show that the problem (1)-(2) has a viscosity subsolution and a viscosity supersolution.

We first show the existence of a viscosity supersolution. If Ω is convex, the Dirichlet problem of Poisson equation

$$\Delta v - \mathrm{tr}\omega(x, v, Dv) = 0, \quad x \in \Omega,$$
$$v = \psi(x), \quad x \in \partial\Omega$$

has a classical solution $\overline{v}$ (see [3], Theorem 15.10). We proceed to show that $\overline{v}$ is a viscosity supersolution of the problem (1)-(2) by contradiction. Suppose $\overline{v}$ is not a viscosity supersolution of (1) in Ω. Then, there exists a function $\varphi \in C^2(\Omega)$, touching $\overline{v}$ from below at $x_0 \in \Omega$ such that

$$[\sigma_k(\mu(D^2\varphi(x_0) - \omega(x_0, \varphi(x_0), D\varphi(x_0))))]^{1/k} > f(x_0, \varphi(x_0), D\varphi(x_0)) > 0.$$

It follows that

$$\mu(D^2\varphi(x_0) - \omega(x_0, \varphi(x_0), D\varphi(x_0))) \in \Gamma_k \subset \Gamma_1.$$

Hence

$$\Delta\varphi(x_0) - \mathrm{tr}(\omega(x_0, \varphi(x_0), D\varphi(x_0))) > 0. \tag{14}$$

On the other hand, since φ touches $\overline{v}$ from below at x_0, we obtain

$$\varphi(x_0) = \overline{v}(x_0), \quad D\varphi(x_0) = D\overline{v}(x_0), \quad D^2\overline{v}(x_0) \geq D^2\varphi(x_0).$$

It follows that $\Delta\varphi(x_0) \leq \Delta\overline{v}(x_0)$ and

$$\Delta\varphi(x_0) - \mathrm{tr}(\omega(x_0, \varphi(x_0), D\varphi(x_0))) \leq \Delta\overline{v}(x_0) - \mathrm{tr}(\omega(x_0, \overline{v}(x_0), D\overline{v}(x_0))) = 0,$$

which contradicts (14).

We proceed to show the existence of a viscosity subsolution. A function $a : \mathbb{R}^n \to \mathbb{R}$ is said to be an affine function if a can be written as

$a(x_1,\cdots,x_n)=c_0+c_1x_1+\cdots+c_nx_n$, with $c_i\in\mathbb{R}, i=1,2,\cdots,n.$

Let

$\mathcal{S}=\{a: a$ is an affine function and $a\le\psi$ on $\partial\Omega\}$.

In view of (10), each element $a\in\mathcal{S}$ is a viscosity subsolution of the (1)-(2). Let $\underline{v}(x)=\sup\{a(x): a\in\mathcal{S}\}$. It is clear that $\underline{v}$ is a viscosity subsolution of (1). We need to show that $\underline{v}=\psi$ on $\partial\Omega$. Indeed, we have $\underline{v}\le\psi$ on $\partial\Omega$. We show that $\underline{v}(\kappa)\ge\psi(\kappa)$ with $\kappa\in\partial\Omega$. Without loss of generality, suppose $\kappa=0$. Then $x_1=0$ is a supporting hyperplane to Ω at 0 and $x_1>0$ for any $x=(x_1,\cdots,x_n)\in\Omega$. By the continuity of ψ, for any $\varepsilon>0$ there exists $\delta>0$ such that $|\psi(x)-\psi(0)|<\varepsilon$ for any $|x|<\delta$, $x\in\partial\Omega$. Since $\partial\Omega$ is strictly convex, there exists $\vartheta>0$ so that

$\bar{\Omega}\cap\{x=(x_1,\cdots,x_n): x_1<\vartheta\}\subset\{x:|x|<\delta\}$.

Let $K:=\min\{\psi(x): x\in\partial\Omega, x_1\ge\vartheta\}$. Then the affine function $a(x)=\psi(0)-\varepsilon-Lx_1$ with $L\ge\max\{(\psi(0)-\varepsilon-K)/\vartheta,0\}$ satisfies $a(0)\ge\psi(0)-\varepsilon$ and $a(x)\le\psi(x)$ with $x\in\partial\Omega$. Since $a\in\mathcal{S}, \underline{v}\ge a$. In particular, $\underline{v}(0)\ge a(0)\ge\psi(0)-\varepsilon$, letting $\varepsilon\to0$ we obtain $\underline{v}(0)\ge\psi(0)$. Thus, $\underline{v}=\psi$ on $\partial\Omega$.

3. CONCLUSION

In this paper we have established several sufficient conditions for the existence and uniqueness of continuous viscosity solutions for Dirichlet problem of the augmented $k-$ Hessian equations with nonsmooth data. We have also proved the $(\omega,k)-$convexity of the viscosity solutions. Our results are extensions of results in Colesanti [2].

Acknowledgements: *This research was funded by Hanoi Pedagogical University 2 foundation for Sciences and Technology Development via grant number: C2020-SP2-05*

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NGHIỆM NHỚT CỦA PHƯƠNG TRÌNH KIỂU K – HESSIAN

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Tóm tắt: *Bài báo này nghiên cứu bài toán Dirichlet đối với phương trình kiểu $k-$ Hessian trong miền bị chặn với dữ kiện không nhất thiết trơn. Chúng tôi giới thiệu khái niệm hàm $(\omega, k)-$ lồi, chỉ ra rằng mọi nghiệm dưới nhớt, nghiệm trên nhớt của bài toán Dirichlet đều là $(\omega, k)-$ lồi, đồng thời chứng minh một số điều kiện đủ về sự tồn tại và tính duy nhất nghiệm nhớt của bài toán Dirichlet đang xét.*

Từ khóa: *Phương trình kiểu $k-$ Hessian, nghiệm nhớt.*

(Ngày Tòa soạn nhận được bài: 25-8-2022; ngày phản biện đánh giá: 26-8-2022; ngày chấp nhận đăng: 29-8-2022)

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