WEAK* FIXED POINT PROPERTY OF FOURIER-STIELTJES ALGEBRA ON COMPACT GROUPS

Bui Ngoc Muoi[1,✉], Tran Thi Thu[1], Nguyen Quoc Tuan[1]

Abstract: *Lau and Mah [3] showed that a Fourier-Stieltjes algebra $B(G)$ on a separable compact group G has the weak* fixed point property, i.e. every nonexpansive mapping on a weak* compact convex subset of $B(G)$ has a fixed point. We extend this result by showing that a similar fixed point property holds for norm continuous and asymptotically nonexpansive mappings.*

Keywords: *Fourier-Stieltjes algebra, semitopological semigroup, fixed point property.*

1. INTRODUCTION

Let K be a non-empty convex subset of a Banach space X. Let $T: K \to K$ be a *nonexpansive* map, namely $\|Tx - Ty\| \leq \|x - y\|$, for all x, y in K. Schauder [5] shows that T has a fixed point if K is norm compact. However, if K is weakly (resp. weak*) compact convex subset of a Banach (resp. dual Banach) space, then T does not necessarily have a fixed point, see [1] for more discussion. We called a dual Banach space X has *weak* fixed point property* if every nonexpansive mapping on a weak* compact convex subset of X has a fixed point.

Let $L^1(G)$ be the group algebra of G associated with the regular left Haar measure $d\lambda$ with convolution product. Define a norm on $L^1(G)$ by

$$\|f\|_* = \sup \|\pi(f)\|,$$

where the supremum is taken over all nondegenerate $*-$ representations $\pi: L^1(G) \to B(H_\pi)$ for some Hilbert space H_π. Let $C^*(G)$ be the completion of $L^1(G)$ with respect to the $\|.\|_*$ norm.

[1] Hanoi Pedagogical University 2

✉ Corresponding Author. Email: buingocmuoi@hpu2.edu.vn

Let $P(G)$ be the set of all positive definite functions on G, i.e. for each function $f \in P(G)$ there is a unitary representation $\pi : G \to B(H_\pi)$ for some Hilbert space H_π and some $\xi \in H_\pi$ such that $f(s) = \langle \pi(s)\xi, \xi \rangle$ for all $s \in G$. Let $B(G)$ be the linear span of $P(G)$. In this way, $B(G)$ becomes a Banach algebra with the pointwise multiplication, and it is the dual space of $C^*(G)$ with the norm defined by

$$\|\varphi\|_{B(G)} = \sup \left\{ \left| \int \varphi(t) f(t) d\lambda(t) \right| : f \in L^1(G), \|f\|_* \le 1 \right\}.$$

A *semitopological semigroup* S is a semigroup with a Hausdorff topology such that the product is separately continuous, i.e., for each fixed $t \in S$, both the maps $s \mapsto ts$ and $s \mapsto ts$ from S into S are continuous. We call S *left reversible* (resp. *right reversible*) if any two right ideals (resp. left ideals) of S always intersect, in other words, for each $s, t \in S$ we have $\overline{sS} \cap \overline{tS} \neq \varnothing$ (resp. $\overline{Ss} \cap \overline{St} \neq \varnothing$). We call S *reversible* if it is both left and right reversible. Examples of reversible semigroup includes topological groups, commutative semitopological semigroups and discrete inverse semigroups.

An *action of* a semitopological semigroup S on a Hausdorff topological space K is a mapping of $S \times K$ into K, denoted by $(s, x) \mapsto s.x$, such that $st\ x = s.\ tx$ for all $s, t \in S$ and $x \in K$. We call the action *continuous* if the mapping $s, x \mapsto T_s x$ is separately continuous. A point $x_0 \in K$ is called a *common fixed point* for S if $s.x_0 = x_0$ for all $s \in S$.

Definition 1.1 ([4]). An action $S \times K \mapsto K$ is called of

(i) asymptotically nonexpansive type if for each $x, y \in K$ and $\varepsilon > 0$, there exist a left ideal I and a right ideal J in S such that

$$\|sx - sty\| \le \|s - ty\| + \varepsilon \quad \text{for all } s \in I \text{ and } t \in J; \tag{1.1}$$

(ii)strongly asymptotically nonexpansive type if for each $x, y \in K$, $\varepsilon > 0$ and each right ideal J of S, there exist a left ideal I of S such that

$$\|sx - sty\| \le \|s - ty\| + \varepsilon \quad \text{for all } s \in I \text{ and } t \in J.$$

A map $T : K \mapsto K$ is called *asymptotically nonexpansive* if for each $x, y \in K$,

$$\lim_k \sup = \|T^k x - T^k y\| \le \|x - y\| \tag{1.2}$$

then is of asymptotically nonexpansive type. We have asymptotic nonexpansiveness is strictly weaker than nonexpansiveness, see [4] for an example. Furthermore, if a map T is asymptotically nonexpansive on K then the action of $S = \{T^k : k \in N\}$ on K is asymptotically nonexpansive type.

Lau and Mah [3] showed that

Theorem 1.2 ([3, Theorem 4.2]). *Let G be a separable compact group. Let K be a non-empty weak* compact convex subset of the Fourier-Stieltjes algebra $B(G)$. Let S be a left reversible semitopological semigroup. Let $S \times K \mapsto K$ be a nonexpansive, norm continuous action of S on K. Then S has a common fixed point in K.*

As a consequence, B(G) has a weak* fixed point property. Indeed, for a nonexpansive mapping T, consider the semigroup S = $\{T^k : k \in N\}$ generated by T. Then S has a common fixed point by Theorem 1.2, hence T has a fixed point. Motivated by Theorem 1.2, we show in this paper fixed point properties for a semigroup of asymptotically nonexpansive mappings. As a consequence B(G) has weak* fixed point property for asymptotically nonexpansive mappings. This provides a variance and generalization of some results in [3].

2. RESEARCH CONTENT

Let K be a nonempty subset of a Banach space X and let $\{D_\lambda : \lambda \in \Delta\}$ be a decreasing net of bounded non-empty subsets of X. For each x $\in$K and $\lambda \in \Delta$, let

$$\begin{aligned} &r_\lambda(x) = \sup\{\|x - y\| : y \in D_\lambda\}, \\ &r(x) = \lim_\lambda r_\lambda(x) = \inf_\lambda r_\lambda(x), \\ &r = \inf\{r(x) : x \in K\}. \end{aligned} \tag{2.3}$$

The set (possibly empty)

$$AC(\{D_\lambda : \lambda \in \Delta\}) = \{x \in K : r(x) = r\}$$

Consists of *asymptotic centers*, and *r* is called the *asymptotic radius*, of $\{D_\lambda : \lambda \in \Delta\}$ with respect to K.

The following lemma arises from the proof of [2, Theorem 3.1].

Lemma 2.1. *Let S be a right reversible semitopological semigroup. Assume $S \times K \to K$ is a separately continuous action of S on a compact convex subset K of a* *locally convex space. Then there exists a subset L_0 of K which is minimal with respect to being nonempty, compact, convex and satisfying the following conditions **(*1)** and **(*2).***

***(*1)** there exists a collection $\Delta = \{\Delta_i : i \in I\}$ of closed subsets of K such that $L_0 = \bigcap \Delta$, and*

***(*2)** for each $x \in L_0$ there is a left ideal $J_i \subseteq S$ such that $J_i.x \subseteq \Delta_i$ for each $i \in I$.*

Furthermore, L_0 contains a subset Y that is minimal with respect to being nonempty, compact, and S-invariant, i.e., $s.Y \subseteq Y$ for all $s \in S$.

Proof. We sketch the arguments in the proof of [2, Theorem 3.1]. By the Zorn's lemma such L_0 always exists. For each $x \in L_0$, let Φ be the collection of all finite intersections of sets in $\{\Delta_i : i \in I\}$. Order Φ by the reverse set inclusion. For any $\alpha = \Delta_1 \bigcap \Delta_2 \bigcap ... \bigcap \Delta_n \in \Phi$, choose left ideals J_i such that $J_i.x \subseteq \Delta_i$ for $i = 1,....,n$. By the right reversibility, there exists $s_\alpha \in \bigcap_{i=1}^n \overline{J_i}$. Thus, S $s_\alpha.x \subseteq \alpha$. Let z be a cluster point of the net $\{s_\alpha.x\}_{\alpha \in \Phi}$.

Then $\overline{Sz}$ is a closed S-invariant subset of L_0. By Zorn's lemma, there exists a minimal subset $Y \subseteq \overline{Sz} \subseteq L_0$ with respect to being nonempty, closed and S-invariant.

The following lemma is crucial for our results.

Lemma 2.2 ([3, Theorem 4.1]). *Let G be a separable compact group. Let K be a nonempty weak* closed convex subset of B(G). Let $\{D_\lambda : \lambda \in \Delta\}$ be a downward directed net of nonempty bounded subsets of K. Then the set of asymptotic centers of $\{D_\lambda : \lambda \in \Delta\}$ with respect to K is nonempty, norm-compact and convex.*

We establish in the following weak* fixed point properties of B(G) that provides a variance of [3] for various nonexpansive mappings.

Theorem 2.3 *Let G be a separable compact group. Let K be a non-empty weak* compact convex subset of A(G). Let $S \times K \to K$ be a norm continuous action of a semitopological semigroup S on K. Assume either*

(i) S is commutative and the action is of asymptotically nonexpansive type, or

(ii)S is reversible and the action is of strongly asymptotically nonexpansive type.

Then S has a common fixed point in K.

Proof. We follow the idea in proving Theorem 1.2 in [3]. For a fixed $z \in k$ and $s \in S$, let $W_s = \overline{sSz}$. Direct S by the order $s \geq t$ if $sS \subset \overline{tS}$. Then $\{W_s : s \in S\}$ is a decreasing net of subsets of K. By Lemma 2.2, the set C of asymptotic center of $\{W_s : s \in S\}$ with respect to K is a non-empty convex norm-compact set.

We show that for each x in C, there is a left ideal I of S such that $Ix \subset C$. Indeed, for each $\varepsilon > 0$, there is a $t \in S$ such that $r_t(x) = \sup\{\|x - y\| : y \in W_t\} \leq r + \frac{\varepsilon}{2}$. Hence $tSz \subset W_t \subset \overline{B}[x, r + \frac{\varepsilon}{2}]$, where $\overline{B}[x, r]$ is the closed ball center at x with radius r. Then

$$\|ts'.z - x\| \leq r + \frac{\varepsilon}{2}, \text{ for all } s' \in S. \tag{2.4}$$

Since the action is asymptotically nonexpansive type, for $x, t.z \in K$ and $\varepsilon > 0$, there are left ideals I and J of S such that

$$\|ss't.z - s.x\| \leq\leq \|s't.z - x\| + \frac{\varepsilon}{2} \leq r + \varepsilon, \text{ for all } s \in I \text{ and } s' \in J,$$

where the second inequality follow from (2.4) and the commutativity of S. Therefore, there exists an $s_1 \in tIJ \subset S$ such that

$$s_1Sz \subset IJtz \subset \overline{B}[s.x, r + \varepsilon].$$

Thus, $W_{s_1} \subset \overline{B}[s.x, r + \varepsilon]$. In other word, $sx \in C$ for all $s \in I$. Hence, $Ix \subset C$.

Similar to the case (ii), for given $x, z \in C, \varepsilon > 0$ and a right ideal tS of S, there is a left ideal I of S such that

$$\|s.x - stx'.z\| \leq \|x - ts'.z\| + \frac{\varepsilon}{2} \leq r + \varepsilon, \quad \text{for all } s \in I, s' \in S.$$

Hence $stSz \subset W_{st} \subset \overline{B}\ s.x, r + \varepsilon$, and thus $s.x \in C$ for all $s \in I$.

Follow Lemma 2.1 and the preceding discussion, there exists a subset L_0 of C which is minimal with respect to being nonempty, norm-compact, convex and satisfying the following conditions.

$*1$ there exists a collection $\Lambda = \ \Lambda_i : i \in I\$ of closed subsets of C such that $L_0 = \cap\Lambda$, and

$*2$ for each $x \in L_0$ there is a left ideal $J_i \subseteq S$ such that $J_i x \subseteq \Lambda_i$; for each $i \in I$. Furthermore, L_0 contains an S-invariant subset $\overline{Su}$ for some $u \in L_0$.

If L_0 contains one point then u is a common fixed point of S. Suppose that L_0 contains more than one point. For each $s \in S$, let $U_s = \overline{sSu}$. Then $U_s : s \in S$ is a decreasing net of subsets of L_0. Then the asymptotic center $AC \quad U_s : s \in S$ with respect to L_0 is a nonempty compact convex proper subset of Lo. Following an approach in [4], we show that $AC \quad U_s : s \in S$ also satisfies properties $*1$ and $*2$.

Let r be the asymptotic radius of $U_s : s \in S$ with respect to L_0. For each $n \in N$, let $C_n = \left\{ x \in L_0 : r \;\; x \;\; \leq r + \dfrac{1}{n} \right\}$. Then C_n is a nonempty closed convex subset of L_0. Moreover

$$AC \quad U_s : s \in S \quad = \bigcap_{n=1}^{\infty} C_n.$$

Let $x \in AC \quad U_s : s \in S$ and consider a fixed C_n. Since $x \in C_{3n}$, we have $r \;\; x \;\; \leq r + \dfrac{1}{3n}$ Hence there exists an $t \in S$ such that

$$r_t \;\; x \;\; = \sup \;\; \|x - y\| : y \in U_t \;\; \leq r + \frac{1}{2n}.$$

From (1.1), there are a left ideal I and a right ideal J of S such that

$$\|sx - ss'u\| \leq \|x - s'u\| + \frac{1}{2n}$$

for all $s \in I$ and $s' \in J$. Take $t_0 \in \overline{J} \cap \overline{tS}$, we can assume $J = t_0 S$ for some $t_0 \in S$ such that $\overline{Ju} = U_{t_0} \subset U_t$ for all $t \in J$. Hence

$$\|sx - sy\| \leq \|x - y\| + \frac{1}{2n} \leq r + \frac{1}{n}$$

for all $s \in I$ and $y \in U_{t_0}$. Thus

$$\|sx - z\| \leq r + \frac{1}{n}$$

for all $s \in I$ and $z \in U_{st_0} = \overline{st_0 Su}$. Hence

$$r\ \ sx\ \le r_{st_0}\ \ sx\ = \sup\ \ \|sx - y\| : y \in U_{st_0}\ \ \le r + \frac{1}{n}$$

for all $s \in I$. In other words, $Ix \subset C_n$. This implies that the proper subset $AC\ \ U_s : s \in S$ of L_0 has also properties $*1$ and $*2$. The contradiction shows that L_0 consists one point and it is a common fixed point of S .

As consequence, $B(G)$ has the weak* fixed point property for asymptotically nonexpansive mappings.

Corollary 2.4. *Let G be a separable compact group, let K be a weak* compact convex subset of B(G). Let T be a norm-continuous and asymptotically nonexpansive map from K into K. Then T has a fixed point in K.*

Proof. Let $S = \ T^n : n \in N$ be the commutative discrete semigroup generated by T .Then the action of S on K is separately norm continuous and asymptotically nonexpansive type. From Theorem 2.3, S has a common fixed point in K, then so is T .

3. CONCLUSION

We have showed that if G is a separable compact group then the Fourier- Stieltjes algebra $B(G)$ has a fixed point property for the semigroup of asymptotically nonexpansive type mappings. As a consequence, $B(G)$ has the weak* fixed point property for asymptotically nonexpansive mappings. This extends some results of Lau and Mah [3] in literature.

Acknowledgements: *This research was also funded by Hanoi Pedagogical University 2 Foudation for Sciences and Technology Development via grant numver HPU2.UT-2021.03*

REFERENCES

1. D. Alspach, A fixed point free nonexpansive map, *Proc. Amer. Math. Soc.* (1981), 423-424.
2. R. D. Holmes and A. T.-M. Lau, Asymptotically nonexpansive actions of topological semigroups and fixed points, *Bull. London Math. Soc*., 3 (1971), 343–347.
3. A. T.-M. Lau and P. Mah, Fixed point properties for Banach algebras associated to locally compact groups, *J. Funct. Anal*., 258 (2010), 357–372.

4. S. Saeidi, F. Golkar and A. M. Forouzanfar, Existence of fixed points for asymptotically nonexpansive type actions of semigroups, J. *Fired Point Theory Appl*., 20 (2018), no. 2, Art. 72.

5. J. Schauder, Der Fixpunktsatz in Funktionalraumen, *Studia Math*., 2 (1930), 171-180.

TÍNH CHẤT ĐIỂM BẤT ĐỘNG YẾU* CỦA ĐẠI SỐ FOURIER-STIELTJES TRÊN NHÓM COMPACT

Bùi Ngọc Mười, Trần Thị Thu,
Nguyễn Quốc Tuấn

Tóm tắt: *Lau và Mah trong [3] đã chỉ ra rằng đại số Fourier-Stieltjes xác định trên một nhóm compact có tính chất điểm bất động yếu*. Tức là, mọi ánh xạ không giãn trên một tập lồi và compact yếu* đều có điểm bất động. Trong bài báo này ta trình bày một mở rộng của tính chất nêu trên. Cụ thể là, ta chỉ ra rằng một kết quả tương tự vẫn đúng cho các ánh xạ xấp xỉ không giãn và liên tục theo chuẩn.*

Từ khóa: *Đại số Fourier-Stieltjes, nửa nhóm topo, tính chất điểm bất động.*

(Ngày Tòa soạn nhận được bài: 23-8-2022; ngày phản biện đánh giá: 26-8-2022; ngày chấp nhận đăng: 30-8-2022)