ON THE STOCHASTIC DOMINATION IN CESÀRO SENSE

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Abstract: In this paper, we study the notion of *stochastically dominated* in Cesàro sense of double arrays of random variables which introduced by Fazekas and Tómács [2]. We also establish the necessary and sufficient conditions for double arrays of random variables which is stochastically dominated in Cesàro sense by a non-negative random variable X.

Keywords: Double array; stochastic domination; random variable; distribution function.

1 Introduction

Let I be a nonempty index set. A family of random variables $\{X_i, i \in I\}$ is said to be *stochastically dominated* by a random variable X if

$$\sup_{i\in I} \mathbb{P}(|X_i| > x) \le \mathbb{P}(|X| > x), \text{ for all } x \in \mathbb{R}. \tag{1.1}$$

We note that many authors use an apparently weaker definition of $\{X_i, i \in I\}$ being *stochastically dominated* by a random variable Y, namely that

$$\sup_{i\in I} \mathbb{P}(|X_i| > x) \le C\mathbb{P}(|Y| > x), \text{ for all } x \in \mathbb{R} \tag{1.2}$$

for some constant $C \in (0, \infty)$, but it is shown by Rosalsky and Thanh [**?**], *inter alia*, that (1.1) and (1.2) are indeed equivalent.

Now we investigate the notion of a double array of independent random variables being *stochastically dominated* in Cesàro by a nonnegative random variable X. This notion was first introduced by Gut [3] for triangular arrays of random variables and extended to multi-dimensional arrays of random variables by Fazekas and Tómács [2]. Let $\{X_{mn}, m \ge 1, n \ge 1\}$ be a double array of random variable. We say that $\{X_{mn}, m \ge 1, n \ge 1\}$ is *stochastically dominated* by a nonnegative random variable X if

$$\sup_{m\ge1,n\ge1} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} \mathbb{P}(|X_{i,j}| > x) \le \mathbb{P}(X > x) \text{ for all } x \in \mathbb{R}. \tag{1.3}$$

We note that Fazekas and Tómács [2] introduced an apparently weaker definition of $\{X_{mn}, m \ge 1, n \ge 1\}$ being *stochastically dominated* by a random variable Y, namely that

$$\sup_{m\ge1,n\ge1} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} \mathbb{P}(|X_{i,j}| > x) \le C\mathbb{P}(|Y| > x) \text{ for all } x \in \mathbb{R} \tag{1.4}$$

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for some constant $C \in (0, \infty)$. Similar to the result of Rosalsky and Thành [4], we will prove in this paper that (1.3) and (1.4) are actually equivalent.

2 Main result

For a double array of random variables $\{X_{m,n}; m \geq 1, n \geq 1\}$, the following theorem characterizes when the function

$$F(x) = 1 - \sup_{m\geq 1, n\geq 1} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{P}(|X_{i,j}| > x), x \in \mathbb{R}$$

is the distribution function of a nonnegative random variable X such that $\{X_{m,n}; m \geq 1, n \geq 1\}$ is stochastically dominated by X in Cesàro sense.

Theorem 2.1. *Let $\{X_{m,n}; m \geq 1, n \geq 1\}$ be a double array of random variables and let*

$$F(x) = 1 - \sup_{m\geq 1, n\geq 1} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{P}(|X_{i,j}| > x), x \in \mathbb{R}.$$

Then F is nondecreasing, right continuous and $\lim_{x\to-\infty} F(x) = 0$. Moreover, F is the distribution function of a nonnegative random variable X if and only if $\lim_{x\to\infty} F(x) = 1$.
In such a case, $\{X_{m,n}; m \geq 1, n \geq 1\}$ is stochastically dominated by X in Cesàro sense, i.e,

$$\sup_{m\geq 1, n\geq 1} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{P}(|X_{i,j}| > x) \leq \mathbb{P}(X > x) \text{ for all } x \in \mathbb{R}.$$

Proof. It is easy to see that F is nondecreasing. Since $\mathbb{P}(|X_{i,j}| > x) = 1$ for all $x < 0, 1 \leq i \leq m, 1 \leq j \leq n$, we have $\lim_{x\to-\infty} F(x) = 0$.

Let $G(x) = \sup_{m\geq 1, n\geq 1} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{P}(|X_{i,j}| > x), x \in \mathbb{R}$. To show that F is right continuous, we will show that G is right continuous; that is, we will show that

$$\lim_{x\to a^+} G(x) = G(a) \text{ for all } a \in \mathbb{R}.$$

Let $\varepsilon > 0$ and let $a \in \mathbb{R}$. Since $G(a) = \sup_{m\geq 1, n\geq 1} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{P}(|X_{i,j}| > a)$, there exists $m_0 \geq 1, n_0 \geq 1$ such that

$$\frac{1}{m_0 n_0} \sum_{i=1}^{m_0} \sum_{j=1}^{n_0} \mathbb{P}(|X_{i,j}| > x) > G(a) - \frac{\varepsilon}{2}.$$

Since the function

$$x \mapsto \frac{1}{m_0 n_0} \sum_{i=1}^{m_0} \sum_{j=1}^{n_0} \mathbb{P}(|X_{i,j}| > x), x \in \mathbb{R}$$

is nonincreasing and right continuous, there exists $\delta > 0$ such that

$$-\frac{\varepsilon}{2} < \frac{1}{m_0 n_0}\sum_{i=1}^{m_0}\sum_{j=1}^{n_0}\mathbb{P}(|X_{i,j}| > x) - \frac{1}{m_0 n_0}\sum_{i=1}^{m_0}\sum_{j=1}^{n_0}\mathbb{P}(|X_{i,j}| > a) \le 0 \text{ for all } x \text{ such that } 0 \le x-a < \delta.$$

Therefore, for x sastisfying $0 \le x - a < \delta$, we have

$$\begin{aligned} G(x) + \varepsilon &= \sup_{m\ge1, n\ge1} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{P}(|X_{i,j}| > x) + \varepsilon \\ &\ge \frac{1}{m_0 n_0}\sum_{i=1}^{m_0}\sum_{j=1}^{n_0}\mathbb{P}(|X_{i,j}| > x) + \varepsilon \\ &> \frac{1}{m_0 n_0}\sum_{i=1}^{m_0}\sum_{j=1}^{n_0}\mathbb{P}(|X_{i,j}| > a) + \frac{\varepsilon}{2} \\ &> G(a) \end{aligned}$$

and so $|G(x) - G(a)| < \varepsilon$. Thus $\lim\limits_{x\to a^+} G(x) = G(a)$. Since F is nondecreasing, right continuous and $\lim\limits_{x\to-\infty} F(x) = 0$, it is the distribution function of a random variable X if and only if

$$\lim_{x\to\infty} F(x) = 1.$$

From $\mathbb{P}(|X_{i,j}| > x) = 1$ for all $1 \le i \le m, 1 \le j \le n$ and $x < 0$, we have $F(x) = 0$ for all $x < 0$. This implies $\mathbb{P}(X > x) = 1 - F(x) = 1$ for all $x < 0$ and so

$$\mathbb{P}(X \ge 0) = \mathbb{P}\left(\bigcap_{k=1}^{\infty}(X > -\frac{1}{k})\right) = 1$$

that is $X \ge 0$ almost surely (a.s.). By definition of F, it is clear that $\{X_{i,j}\ ;\ 1 \le i \le m, 1 \le j \le n\}$ is stochastically dominated by X in Cesàro sense. □

Before establishing the equivalence between the definitions of stochastic domination given in (1.3) and (1.4), we present the following simple lemma. This is Lemma 2.3 of Rosalsky and Thành [4].

Lemma 2.2. *Let $g : [0,\infty) \to [0,\infty)$ be a measurable function with $g(0) = 0$ which is bounded on $[0, A]$ and differentiable on $[A,\infty)$ for some $A \ge 0$. If ξ is a nonnegative random variable, then*

$$\mathrm{E}(g(\xi)) = \mathbb{E}(g(\xi)\mathbf{1}(\xi \le A)) + g(A) + \int_A^{\infty} g'(x)\mathbb{P}(\xi > x)\mathrm{d}x. \tag{2.1}$$

The next theorem establishes the equivalence between the definitions of stochastic domination given (1.3) and (1.4).

Theorem 2.3. *Let $\{X_{m,n}, m \geq 1, n \geq 1\}$ be a double array of random variables. Then there exists a nonnegative random variable X satisfying* (1.3) *if and only if there exist a nonnegative random variable Y and a constant $C \in (0,\infty)$ satisfying* (1.4)*. Moreover,*

(i) *if $g : [0,\infty) \to [0,\infty)$ is a measurable function with $g(0) = 0$ which is bounded on $[0, A]$ and differentiable on $[A,\infty)$ for some $A \geq 0$*

or

(ii) *if $g : [0,\infty) \to [0,\infty)$ is a continuous function which is eventually nondecreasing with $\lim_{x\to\infty} g(x) = \infty$,*

then the condition $\mathbb{E}(g(Y)) < \infty$ where Y is as in (1.4) *implies that $\mathbb{E}(g(X)) < \infty$ where X is as in* (1.3)*.*

Proof. Firstly, we need to show that there exists a nonnegative random variable X satisfying (1.3) if and only if there exist a nonnegative random variable Y and a constant $C \in (0,\infty)$ satisfying (1.4). The necessity half is immediate by taking $Y = X$ and $C = 1$. Conversely, assume that there exists a nonnegative random variable Y and a constant $C \in (0,\infty)$ satisfying (1.4). Then

$$\lim_{x\to\infty} \sup_{m\geq 1, n\geq 1} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{P}(|X_{i,j}| > x) \leq C \lim_{x\to\infty} \mathbb{P}(Y > x) = 0$$

and so by Theorem 2.1, there exists a nonnegative random variable X with the distribution function

$$F(x) = 1 - \sup_{m\geq 1, n\geq 1} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{P}(|X_{i,j}| > x), x \in \mathbb{R} \tag{2.2}$$

such that (1.3) holds.

Secondly, we prove that if $g : [0,\infty) \to [0,\infty)$ is a measurable function with $g(0) = 0$ which is bounded on $[0, A]$ and differentiable on $[A,\infty)$ for some $A \geq 0$ then the condition $\mathrm{E}(g(Y)) < \infty$ where Y is as in (1.4) implies that $\mathbb{E}(g(X)) < \infty$ where X is as in (1.3). By (1.4) and (2.2),

$$\mathbb{P}(X > x) = \sup_{m\geq 1, n\geq 1} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{P}(|X_{i,j}| > x) \leq C\mathbb{P}(Y > x), x \in \mathbb{R}. \tag{2.3}$$

By Lemma 2.2 and (2.3), we have

$$\begin{aligned}
\mathbb{E}(g(X)) &= \mathbb{E}(g(X)\mathbf{1}(X \leq A)) + g(A) + \int_A^\infty g'(x)\mathbb{P}(X > x)\mathrm{d}x \\
&\leq \mathbb{E}(g(X)\mathbf{1}(X \leq A)) + g(A) + C\int_A^\infty g'(x)\mathbb{P}(Y > x)\,\mathrm{d}x \\
&\leq C' + C\mathbb{E}(g(Y))
\end{aligned}$$

where C' is a finite positive constant. Thus the condition $\mathbb{E}\left(g\left(Y\right)\right) < \infty$ implies that $\mathbb{E}(g(X)) < \infty$.

Lastly, we now prove that if $g : [0, \infty) \to [0, \infty)$ is a continuous function which is eventually nondecreasing with $\lim\limits_{x\to\infty} g(x) = \infty$, then the condition $\mathbb{E}\left(g\left(Y\right)\right) < \infty$ where Y is as in (1.4) implies that $\mathbb{E}(g(X)) < \infty$ where X is as in (1.3). Let $A > 0$ be such that $g(A) > 0$ and g is nondecreasing on $[A, \infty)$. Let $B = \sup\limits_{0\le x\le A} g(x)$. Then $B < \infty$ since g is continuous. Let

$$h(x) = \begin{cases} \dfrac{xg(A)}{A} & \text{if } 0 \le x < A, \\ g(x) & \text{if } x \ge A. \end{cases}$$

and

$$h^{-1}(t) = \inf\{x \ge 0 : h(x) \ge t\}, t \ge 0.$$

Note that for all $t \ge 0$ and $x \ge 0, t \le h(x)$ if and only if $h^{-1}(t) \le x$. It is easy to see that (2.3) ensures that

$$\mathbb{P}(X \ge x) \le C\mathbb{P}\left(Y \ge x\right), x \in \mathbb{R}. \tag{2.4}$$

Now

$$\begin{aligned}
\mathbb{E}(g(X)) &= \int_0^\infty \mathbb{P}(g(X) \ge x)\mathrm{d}x \\
&= \int_0^\infty \mathbb{P}(g(X) \ge x, X < A)\mathrm{d}x + \int_0^\infty \mathrm{P}(g(X) \ge x, X \ge A)\mathrm{d}x \\
&= \int_0^B \mathbb{P}(g(X) \ge x, X < A)\mathrm{d}x + \int_0^\infty \mathbb{P}(h(X) \ge x, X \ge A)\mathrm{d}x \\
&\le B + \int_0^\infty \mathbb{P}\left(X \ge h^{-1}(x)\right)\mathrm{d}x \\
&\le B + \int_0^\infty C\mathbb{P}\left(Y \ge h^{-1}(x)\right)\mathrm{d}x \quad \text{(by (2.4))} \\
&= B + \int_0^\infty C\mathbb{P}\left(h\left(Y\right) \ge x\right)\mathrm{d}x \\
&= B + C\mathbb{E}\left(h\left(Y\right)\right) \\
&= B + C\mathbb{E}\left(h\left(Y\right)\mathbf{1}\left(Y < A\right)\right) + C\mathbb{E}\left(h\left(Y\right)\mathbf{1}\left(Y \ge A\right)\right) \\
&\le B + Ch(A) + C\mathbb{E}\left(g\left(Y\right)\right).
\end{aligned}$$

Thus the condition $\mathbb{E}\left(g\left(Y\right)\right) < \infty$ implies that $\mathbb{E}(g(X)) < \infty$. □

The following theorem shows that uniformly bounded moment type conditions on a double array of random variables $\{X_{m,n}, m \ge 1, n \ge 1\}$ can guarantee the stochastic domination in Cesàro sense. Throughout the rest of the paper, for $x \ge 0$, we let $\log(x)$ denote $\ln(\max\{e, x\})$ where ln is the natural logarithm.

Theorem 2.4. *Let $\{X_{m,n}, m \ge 1, n \ge 1\}$ be a double array of random variables.*

(i) *Let* $g:[0,\infty)\to[0,\infty)$ *be a nondecreasing function with* $\lim_{x\to\infty} g(x)=\infty$. *If*

$$\sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\left(g(|X_{i,j}|)\right)<\infty, \tag{2.5}$$

then there exists a nonnegative random variable X *with distribution function*

$$F(x)=1-\sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{P}\left(g(|X_{i,j}|)>x\right), x\in\mathbb{R}$$

such that $\{X_{m,n}, m\ge1, n\ge1\}$ *is stochastically dominated by* X *in Cesàro sense.*

(ii) *If*

$$\sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}(|X_{i,j}|^p)<\infty \quad \textit{for some } p>0, \tag{2.6}$$

then there exists a nonnegative random variable X *with distribution function*

$$F(x)=1-\sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{P}(|X_{i,j}|>x),\ x\in\mathbb{R}$$

such that $\{X_{m,n}; m\ge1, n\ge1\}$ *is stochastically dominated by* X *in Cesàro sense, and*

$$\mathbb{E}(X^p\log^{-1-\varepsilon}(X))<\infty \ \textit{for all } \varepsilon>0. \tag{2.7}$$

(iii) *If*

$$\sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}(|X_{i,j}|^p\log^{1+\varepsilon}|X_{i,j}|)<\infty \quad \textit{for some } p>0 \textit{ and for some } \varepsilon>0, \tag{2.8}$$

then there exists a nonnegative random variable X *with distribution function*

$$F(x)=1-\sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{P}(|X_{i,j}|>x),\ x\in\mathbb{R},$$

such that $\{X_{m,n}; m\ge1, n\ge1\}$ *is stochastically dominated by* X *in Cesàro sense, and*

$$\mathbb{E}(X^p)<\infty. \tag{2.9}$$

Proof. **(i)** By the monotonicity of g and the Markov inequality, we have for all large x

$$\sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{P}\left(|X_{i,j}|>x\right)\le\frac{1}{g(|x|)}\sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\left(g(|X_{i,j}|)\right).$$

Thus by (2.5) and $\lim_{x\to\infty} g(|x|) = \infty$, we have

$$\lim_{x\to\infty} \sup_{m\geq 1, n\geq 1} \frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbb{P}\left(|X_{i,j}| > x\right) \leq \lim_{x\to\infty} \frac{1}{g(|x|)} \sup_{m\geq 1, n\geq 1} \frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbb{E}\left(g(|X_{i,j}|)\right) = 0.$$

Therefore, we have

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \left(1 - \sup_{m\geq 1, n\geq 1} \frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbb{P}\left(g(|X_{i,j}|) > x\right)\right) = 1.$$

Then by applying Theorem 2.1, we get that $\{X_{m,n}, m \geq 1, n \geq 1\}$ is stochastically dominated by a nonnegative random variable X in Cesàro sense with distribution function

$$F(x) = 1 - \sup_{m,n\geq 1} \frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbb{P}\left(|X_{i,j}| > x\right).$$

This completes the proof of (i).

(ii) It follows from (2.5) that (2.6) holds with $g(x) = x^p,\ x \geq 0$. Then by Part (i), $\{X_{m,n}; m \geq 1, n \geq 1\}$ is stochastically dominated by a nonnegative random variable X in Cesàro sense with distribution function

$$F(x) = 1 - \sup_{m\geq 1, n\geq 1} \frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbb{P}(|X_{i,j}| > x),\ x \in \mathbb{R}.$$

Let $\varepsilon > 0$ and let

$$h(x) = x^p \log^{-1-\varepsilon}(x),\ x \geq 0.$$

Then, we have

$$h'(x) = \frac{x^{p-1}\log^{\varepsilon}(x)(p\log(x) - (1+\varepsilon))}{\log^{2+2\varepsilon}(x)} \leq p x^{p-1} \log^{-1-\varepsilon}(x),\ x > e. \tag{2.10}$$

By Lemma 2.2, (2.10), the Markov inequality and (2.6), we have

$$\begin{aligned}\mathbb{E}(h(X)) &= \mathbb{E}(h(X)\mathbf{1}(X \leq e)) + h(e) + \int_e^{\infty} h'(x)\mathbb{P}(X > x)dx \\ &\leq 2e^p + \int_e^{\infty} p x^{p-1}\log^{-1-\varepsilon}(x)\mathbb{P}(X > x)dx \\ &= 2e^p + \int_e^{\infty} p x^{p-1}\log^{-1-\varepsilon}(x) \sup_{m\geq 1, n\geq 1} \frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbb{P}(|X_{i,j}| > x)dx\end{aligned}$$

$$\le 2e^p + \int_e^\infty px^{-1}\log^{-1-\varepsilon}(x) \sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n \mathbb{E}(|X_{i,j}|^p)dx$$
$$= 2e^p + p\sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n \mathbb{E}(|X_{i,j}|^p)\int_e^\infty x^{-1}\log^{-1-\varepsilon}(x)dx$$
$$< \infty.$$

The proof of (ii) is completed.

(iii) It follows from (2.5) that (2.8) holds with $g(x) = x^p\log^{1+\varepsilon(x)},\ x > 0$. Then by Part (i), $\{X_{m,n}; m \ge 1, n \ge 1\}$ is stochastically dominated by a nonnegative random variable X in Cesàro sense with distribution function

$$F(x) = 1 - \sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n \mathbb{P}(|X_{i,j}| > x),\ x \in \mathbb{R}.$$

Let $\varepsilon > 0$ and let $h(x) = x^p,\ x \ge 0$. Then, we have

$$h'(x) = px^{p-1},\ x \le 0. \tag{2.11}$$

By Lemma 2.2, (2.11), the Markov inequality and (2.8), we have

$$\mathbb{E}(h(X)) = \mathbb{E}(h(X)\mathbf{1}(X \le e)) + h(e) + \int_e^\infty h'(x)\mathbb{P}(X > x)dx$$
$$\le 2e^p + \int_e^\infty px^{p-1}\mathbb{P}(X > x)dx$$
$$= 2e^p + \int_e^\infty px^{p-1}\sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n \mathbb{P}(|X_{i,j}| > x)dx$$
$$\le 2e^p + \int_e^\infty px^{-1}\sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n \mathbb{E}(|X_{i,j}|^p.\log^{1+\varepsilon}(|X_{i,j}|)).\log^{-1-\varepsilon}(x)\ dx$$
$$= 2e^p + p\sup_{m\ge1,n\ge1}\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n \mathbb{E}(|X_{i,j}|^p.\log^{1+\varepsilon}(|X_{i,j}|))\int_e^\infty x^{-1}\log^{-1-\varepsilon}(x)\ dx$$
$$< \infty.$$

The proof of (iii) is completed.

□

The notion of stochastic domination in Cesàro sense of double arrays of random variables has been used in studying laws of large numbers. We refer the reader to [2], [4-7] and references therein for this research direction. The result we achieve in this paper maybe useful for further studies.

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TÓM TẮT

VỀ ĐIỀU KIỆN BỊ CHẶN NGẪU NHIÊN THEO NGHĨA CESÀRO

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Trong bài báo này, chúng tôi nghiên cứu về khái niệm *bị chặn ngẫu nhiên* theo nghĩa Cesàro của mảng kép các biến ngẫu nhiên. Khái niệm này đã được giới thiệu bởi Fazekas and Tómács [2]. Chúng tôi cũng đã thiết lập được điều kiện cần và đủ để mảng kép các biến ngẫu nhiên bị chặn ngẫu nhiên theo nghĩa Cesàro bởi một biến ngẫu nhiên không âm X.

Từ khóa: Mảng kép; bị chặn ngẫu nhiên; biến ngẫu nhiên; hàm phân phối.