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Research Article

A GLOBAL AND POINTWISE GRADIENT ESTIMATE FOR SOLUTIONS TO DOUBLE-PHASE PROBLEMS IN ORLICZ SPACES

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ABSTRACT

The aim of this paper is twofold. Firstly, we give a global estimate of the Calderón-Zygmund type for solutions to double-phase problems in Orlicz spaces via maximal fractional functions. In this study, we employ the approach based on a generalized good-λ technique developed by Tran and Nguyen (2019), where regularity results are preserved under the fractional maximal operator. This operator is notable for its role in evaluating the oscillation of functions, and there is a close relation between this operator and the Riesz potential. Secondly, we present a pointwise estimate of the Riesz potential as a consequence of the first result.

Keywords: Double-phase problems; Orlicz spaces; gradient estimates; Riesz potential; fractional maximal functions

1. Introduction

In this article, we focus our study on the quasilinear elliptic equations with a zero Dirichlet boundary condition as described below

$$\begin{cases} \operatorname{div}\left(\mathcal{A}(x,\nabla u)\right) = \operatorname{div}\left(\mathcal{B}(x,\mathbf{F})\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

in which Ω is an open bounded subset of $\mathbb{R}^n$ $(n \geq 2)$ and the datum $\mathbf{F}:\Omega \to \mathbb{R}^n$ is a vector field. The operators $\mathcal{A},\mathcal{B}:\Omega \to \mathbb{R}^n$ are given such that $\mathcal{A}$ is measurable concerning the first variable and is differentiable with respect to any non-zero second variable while $\mathcal{B}$ is a Carathéodory function. Moreover, they satisfy the following conditions:

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$$\begin{cases} \|\mathcal{A}(x,y)\| + |\partial_y \mathcal{A}(x,y)||y| + |\mathcal{B}(x,y)| \le L\left(|y|^{p-1} + a(x)|y|^{q-1}\right); \\ \gamma\left(|y|^{p-2} + a(x)|y|^{q-2}\right)|z|^2 \le \langle \partial\mathcal{A}(x,y)z, z\rangle ; \\ |\mathcal{A}(x_1,y) - \mathcal{A}(x_2,y)| \le L|a(x_1) - a(x_2)||y|^{q-1}, \end{cases} \tag{A1}$$

with a fixed constant $0<\gamma<L<\infty$ for all non-zero $y,z\in\mathbb{R}^n$ and $x,x_1,x_2\in\Omega$. The function $a:\Omega\to[0,\infty)$ and parameters p, q in (A1) satisfy the following assumptions:

$$0 \le a(.) \in C^{0,\kappa},\ \kappa\in(0,1]; \tag{A2}$$

$$1<\frac{q}{p}\le\left(1+\frac{\kappa}{n}\right);\ p>1; \tag{A3}$$

Problem (P) described above is also known as a double-phase problem, and it is solved together with the main conditions (A1)-(A3). It is a more general form of the following (p,q)-Laplace problem

$$-\mathrm{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = -\mathrm{div}\left(|\mathbf{F}|^{p-2}\mathbf{F} + a(x)|\mathbf{F}|^{q-2}\mathbf{F}\right), \tag{1.2}$$

which is the Euler-Lagrange equation resulting from the energy functional

$$v \mapsto \mathcal{F}(v,\Omega) - \int_\Omega \left\langle |\mathbf{F}|^{p-2}\mathbf{F} + a(x)|\mathbf{F}|^{q-2}\mathbf{F}, \nabla v\right\rangle dx \ ,$$

where $\mathcal{F}(v,\Omega) := \int_\Omega \left(|\nabla v|^p + a(x)|\nabla v|^q\right)dx$ is called double-phase functional.

One of the main concerns when studying problem (P) is the regularity of the weak solutions, which has attracted the interest of several researchers in recent years. Some interesting regularity results will be briefly discussed related to solutions of equation (1.2) or some look-alike non-uniformly elliptic equations. In particular, we are interested in the global or local gradient estimates of the Calderón-Zygmund type of the solutions in different functional spaces. The first results were given by Colombo and Mingione (2016), who investigated the local estimate of the distributional solutions to (1.2). Specifically, they prove that the relation

$$|\mathbf{F}|^p + a(x)|\mathbf{F}|^q \in L^\gamma_{\mathrm{loc}}(\Omega) \Rightarrow |\nabla u|^p + a(x)|\nabla u|^q \in L^\gamma_{\mathrm{loc}}(\Omega) \tag{1.3}$$

is satisfied by any $\gamma\in[1,\infty)$ when $\frac{q}{p}<1+\frac{\kappa}{n}$. In addition, they provided results when $q\le p+\kappa$ and extended those results to the vectorial case. Byun and Oh (2017) improved those results to include the boundary in the case $\partial\Omega$ is a C^{1,κ^+} subset for some $\kappa^+\in[\kappa,1]$. They presented the global estimates in Lebesgue space L^γ . An additional contribution to the study of problem (1.2) was brought into play by Filippis and Mingione (2019), who handled the tricky borderline case $\frac{q}{p}=1+\frac{\kappa}{n}$ and confirmed the validity of the strong relation (1.3) in this case. Following those results, Tran and Nguyen (2021) carried on the regularity study in Lorentz spaces and stated the global gradient estimates for weak solutions to (1.2) via fractional maximal operators in the form

$$\left\|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)\right\|_{L^{s,t}(\Omega)} \le C\left\|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\mathbf{F}\right)\right)\right\|_{L^{s,t}(\Omega)}, \tag{1.4}$$

where $\mathcal{H}$ is defined as below

$$\mathcal{H}(x,y)=|y|^{p}+a(x)|y|^{q},\ x\in\Omega,\ y\in\mathbb{R}^{n}. \tag{1.5}$$

The good-λ technique has been employed in their proof to obtain interesting regularity results in the domain, including the boundary.

The important and appealing results described above have inspired us to extend the study to wider functional spaces. Here, our approach is based on a generalized good-λ technique developed by Tran and Nguyen (2019), where regularity results are preserved under a fractional maximal function $\mathbf{M}_{\beta}$. We shall investigate two main results in this paper. Firstly, we establish the global Calderón-Zygmund-type inequality for the gradient of weak solutions to quasilinear elliptic equations (P) in Orlicz spaces in terms of fractional maximal functions. The second one is a pointwise estimate for the Riesz potential. The structure of the remaining of this article is as follows. Section 2 presents the notations used in this article and gives some preliminary results concerning our proof. Finally, the main results are presented and proved in section 3.

2. Notation and preliminaries

In this section, we briefly introduce some notations, definitions, properties and useful results that will be used throughout the article.

In what follows, we shall assume that the domain $\Omega\subset\mathbb{R}^{n}$ with $n\ge 2$ is open and bounded. The notation $B_{R}\left(x\right)$ stands for an open ball centered at x with the radius $R>0$; that is, the set of all the points $\left\{y\in\mathbb{R}^{n}:|y-x|<R\right\}$. We write $|A|$, when there is no misunderstanding, for Lebesgue measure of a measurable set $A\subset\mathbb{R}^{n}$. For the sake of simplicity, we denote by data the set of parameters arising from assumptions (A1)-(A3) that controls problem (P) under consideration. More specifically, data consists of $n,p,q,\kappa,\gamma,L,\|a\|_{L^{\infty}},[a]_{\kappa},\|\mathcal{H}(\cdot,\nabla u)\|_{L^{1}}$. Finally, we utilize C to denote a universal constant, which may be different from line to line. The dependencies of C on specific parameters will be emphasized in parentheses.

The main result of our article is obtained in Orlicz space. This functional space is defined and has basic properties as stated below.

Definition 2.1. (Young function)

A Young function $\Phi:[0,\infty)\to[0,\infty)$ is a convex and increasing function that satisfies two following limit relations

$$\lim_{\mu\to 0^+}\frac{\Phi(\mu)}{\mu}=0, \qquad \lim_{\mu\to\infty}\frac{\Phi(\mu)}{\mu}=\infty. \tag{2.1}$$

Lemma 2.2. (Hasto, 2019, Lemma 2.2.7)

Let Φ be a Young function; then the following two statements are equivalent:

a) There exists $\tau_1\geq 2$ such that $\Phi(2\mu)\leq\tau_1\Phi(\mu)$ for all $\mu\geq 0$. (2.2)

b) There exist two positive constants K_1 and p_1 such that $\Phi(a\mu)\leq K_1 a^{p_1}\Phi(\mu)$ for any $a>1$ and $\mu>0$. (2.3)

Definition 2.3. (Orlicz space)

Let Φ be a Young function satisfying (2.2). The Orlicz class $\mathrm{O}^{\Phi}(\Omega)$ is specified to be the set of all real-valued, measurable functions f defined on Ω meeting the condition

$$\int_\Omega \Phi\left(|f(x)|\right)dx<\infty.$$

The smallest linear space containing $\mathrm{O}^{\Phi}(\Omega)$, equipped with the norm

$$\|f\|_{L^{\Phi}(\Omega)}=\inf\left\{\tau>0:\int_\Omega\Phi\left(\frac{|f(x)|}{\tau}\right)dx\leq 1\right\},$$

is called the Orlicz space and is denoted as $L^{\Phi}(\Omega)$.

We next state the definition of a solution to problem (P) in the distributional sense.

Definition 2.4. (Distributional solution)

A function $u\in W_0^{1,1}(\Omega)$ satisfying the following condition for every $\varphi\in C_0^{\infty}(\Omega)$

$$\int_\Omega\left\langle\mathcal{A}(x,\nabla u),\nabla\varphi\right\rangle dx=\int_\Omega\left\langle\mathcal{B}(x,\mathbf{F}),\nabla\varphi\right\rangle dx, \tag{2.4}$$

is called a distributional solution to (P) under conditions (A1), (A2), and (A3).

Lemma 2.5. (Byun and Oh, 2017, Proposition 3.5)

Suppose $u\in W_0^{1,1}(\Omega)$ is a distributional solution of (P) under conditions (A1), (A2), and (A3) satisfying $\mathcal{H}(x,\nabla u),\mathcal{H}(x,\mathbf{F})\in L^1(\Omega)$. Then the following variational formula holds for every test function $\varphi\in W_0^{1,1}(\Omega)$ such that $\mathcal{H}(x,\nabla\varphi)\in L^1(\Omega)$

$$\int_\Omega\left\langle\mathcal{A}(x,\nabla u),\nabla\varphi\right\rangle dx=\int_\Omega\left\langle\mathcal{B}(x,\mathbf{F}),\nabla\varphi\right\rangle dx. \tag{2.5}$$

Next, we introduce the doubling property of weights used throughout the article.

Definition 2.6. (Muckenhoupt weights) Let a weight $\omega: \mathbb{R}^n \to [0,\infty)$ be a locally integrable function, we say that $\omega \in A_p$ if one has

$$[\omega]_{A_p} = \sup_{B_r(x)} \left(\frac{1}{|B_r(x)|} \int_{B_r(x)} \omega(y)\,dy \right) \left(\frac{1}{|B_r(x)|} \int_{B_r(x)} \omega(y)^{-\frac{1}{p-1}}\,dy \right)^{p-1} < \infty, \quad \text{when}$$

$1 < p < \infty$

$$[\omega]_{A_1} = \sup_{B_r(x)} \left(\frac{1}{|B_r(x)|} \int_{B_r(x)} \omega(y)\,dy \right) \sup_{y \in B_r(x)} \frac{1}{\omega(y)} < \infty, \text{ when } p = 1$$

and there exist constants $C_0, \nu > 0$ satisfying

$$\omega(A) \le C_0 \left(\frac{|A|}{|B|} \right)^{\nu} \omega(B), \text{ when } p = \infty, \tag{2.6}$$

for every ball $B \subset \mathbb{R}^n$ and all measurable subsets $A \subset B$, where $\omega(A) := \int_A \omega(x)dx.$ In this case, we denote $[\omega]_{A_\infty} = (C_0, \nu).$

Definition 2.7. (Weighted Lorentz spaces) Let $t \in (0,\infty)$, $s \in (0,\infty]$ and $\omega \in A_\infty$, the weighted Lorentz space $L^{t,s}_\omega(\Omega)$ is the set of all Lebesgue measurable functions f defined on Ω whose norm satisfying $\|f\|_{L^{t,s}_\omega(\Omega)} < +\infty$, where

$$\|f\|_{L^{t,s}_\omega(\Omega)} := \left[t \int_0^\infty \lambda^s \omega\left(\{x \in \Omega : |f(x)| > \lambda\}\right)^{\frac{s}{t}} \frac{d\lambda}{\lambda} \right]^{\frac{1}{s}}, \text{ if } s < \infty, \tag{2.7}$$

and

$$\|f\|_{L^{t,\infty}_\omega(\Omega)} := \sup_{\lambda > 0} \lambda \omega\left(\{x \in \Omega : |f(x)| > \lambda\}\right)^{\frac{1}{t}} \quad \text{if } s = \infty. \tag{2.8}$$

We now also look at the definition of the fractional maximal function.

Definition 2.8. (Fractional maximal operator) Let β be a real number in $[0,n]$, the fractional maximal operator $\mathbf{M}_\beta$ of f is given by

$$\mathbf{M}_\beta f(\xi) = \sup_{\rho > 0} \rho^{\beta - n} \int_{B_\rho(\xi)} |f(y)|\,dy, \quad \xi \in \mathbb{R}^n, \tag{2.9}$$

where $f \in L^1_{loc}(\mathbb{R}^n)$. When $\beta = 0$, the Hardy-Littlewood maximal function $\mathbf{M}$ is defined as

$$\mathbf{M}f(\xi) = \sup_{\rho > 0} \rho^{-n} \int_{B_\rho(\xi)} |f(y)|\,dy, \quad \xi \in \mathbb{R}^n. \tag{2.10}$$

Lemma 2.9. (Tran and Nguyen, (2021), Theorem 4.3) Suppose that Ω is an open bounded domain in $\mathbb{R}^n$ such that $\partial\Omega$ belongs to C^{1,κ^+} class for some $\kappa^+ \in [\kappa, 1]$. Let $u \in W_0^{1,1}(\Omega)$ be a distributional solution to (P) under conditions (A1), (A2), and (A3) satisfying $\mathcal{H}(x,\nabla u), \mathcal{H}(x,\mathbf{F}) \in L^1(\Omega)$. Then for any $\beta \in [0,n)$ and $a \in \left(0, 1-\frac{\beta}{n}\right)$, there exist $\varepsilon_0 = \varepsilon_0(n,\beta,a) \in (0,1)$, $b = b(\beta,a) \geq 1$, and a constant $C = C(data, \Omega, \beta, a) > 0$ such that the estimate

$$\left|\left\{x \in \Omega : \mathbf{M}_\beta(\mathcal{H}(x,\nabla u)) > \varepsilon^{-a}\lambda; \mathbf{M}_\beta(\mathcal{H}(x,\mathbf{F})) \leq \varepsilon^b \lambda\right\}\right| \leq C\varepsilon \left|\left\{x \in \Omega : \mathbf{M}_\beta(\mathcal{H}(x,\nabla u)) > \lambda\right\}\right| \quad (2.11)$$

holds for any $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$.

Remark: Results obtained in Tran and Nguyen (2021) were described in the presence of $\mathbf{MM}_\beta$ operator. However, Nguyen and Tran (2020) showed that the good-λ type bound has been improved with $\mathbf{M}_\beta$. Therefore, in (2.11), we state the good-λ result using $\mathbf{M}_\beta$ (See Nguyen & Tran (2020) for further details).

Definition 2.10. (Riesz potential) If $0 < \alpha < n$, then the Riesz potential $I_\alpha f$ of a locally integrable function f on $\mathbb{R}^n$ is a function defined by

$$(I_\alpha f)(z) = \int_{\mathbb{R}^n} \frac{f(y)}{|z-y|^{n-\alpha}} dy. \quad (2.12)$$

3. Main results

Now we state and prove the global gradient estimate in Orlicz spaces via fractional maximal function.

Theorem 3.1. (Global estimate in Orlicz spaces) Let $\beta \in [0,n)$ and let (P) be a problem as defined in section 1 under conditions (A1), (A2), and (A3) on an open bounded domain Ω. Assume that $\partial\Omega$ belongs to C^{1,κ^+} class for some $\kappa^+ \in [\kappa,1]$. Suppose that $u \in W_0^{1,1}(\Omega)$ is a distributional solution to (P) with a given data satisfying $\mathcal{H}(x,\nabla u)$, $\mathcal{H}(x,\mathbf{F}) \in L^1(\Omega)$.

Let $\omega \in A_\infty$ and denote $(C_0, \nu) = [\omega]_{A_\infty}$. Then for $t \in (0,\infty)$ and $0 < s \leq \infty$, there exists a constant $C^* = C^*\left(data, \Omega, t, s, \beta, [\omega]_{A_\infty}\right) > 0$ such that

$$\left\|\mathbf{M}_\beta(\mathcal{H}(x,\nabla u))\right\|_{L_\omega^{t,s}(\Omega)} \leq C^* \left\|\mathbf{M}_\beta(\mathcal{H}(x,\mathbf{F}))\right\|_{L_\omega^{t,s}(\Omega)}. \quad (3.1)$$

Moreover, if $K : \mathbb{R}^+ \to \mathbb{R}^+$ is a Young function and $K(2z) \leq cK(z)$ for all $z \geq 0$ with a constant $c > 0$, then

$$\int_\Omega K\left(\mathbf{M}_\beta(\mathcal{H}(x,\nabla u))\right)\omega(x)dx \leq C^{**} \int_\Omega K\left(\mathbf{M}_\beta(\mathcal{H}(x,\mathbf{F}))\right)\omega(x)dx, \quad (3.2)$$

where the constant $C^{**} = C^{**}\left(data, \Omega, \beta, c, [\omega]_{A_\infty}\right)$.

Proof. First of all, we prove (3.1) for the case $0 < s < \infty$.

For every $t \in (0,\infty)$ and $s \in (0,\infty)$, let us fix $0 < a < \min\left\{1 - \frac{\beta}{n}, \frac{\nu}{t}\right\}$. By virtue of Lemma 2.9, one can find $\varepsilon_0 = \varepsilon_0(n,\beta,a) \in (0,1)$, $b = b(\beta,a) \geq 1$, and a positive constant $C = C(data, \Omega, \beta, a)$ such that the estimate

$$\left|\left\{x \in \Omega : \mathbf{M}_\beta\left(\mathcal{H}(x,\nabla u)\right) > \varepsilon^{-a}\lambda; \mathbf{M}_\beta\left(\mathcal{H}(x,\mathbf{F})\right) \leq \varepsilon^b \lambda\right\}\right| \leq C\varepsilon \left|\left\{x \in \Omega : \mathbf{M}_\beta\left(\mathcal{H}(x,\nabla u)\right) > \lambda\right\}\right| \quad \text{(H)}$$

holds for any $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$.

Let $A = \left\{x \in \Omega : \mathbf{M}_\beta\left(\mathcal{H}(x,\nabla u)\right) > \varepsilon^{-a}\lambda\right\}$, $B = \left\{x \in \Omega : \mathbf{M}_\beta\left(\mathcal{H}(x,\mathbf{F})\right) > \varepsilon^b\lambda\right\}$, $D = \left\{x \in \Omega : \mathbf{M}_\beta\left(\mathcal{H}(x,\nabla u)\right) > \lambda\right\}$ then $A \subset D$ for ε small enough, and the condition (H) says that

$$\left|A \cap B^c\right| \leq C\varepsilon |D|. \tag{3.3}$$

We have

$$\begin{aligned} A = A \cap \Omega = A \cap \left(B \cup B^C\right) &= (A \cap B) \cup \left(A \cap B^C\right) \\ &\subset B \cup \left(A \cap B^C\right). \end{aligned} \tag{3.4}$$

From the definition of ω, $\omega(E) = \int_E \omega(x)dx$, it implies from (3.4) that

$$\omega(A) \leq \omega\left[B \cup \left(A \cap B^C\right)\right] = \omega(B) + \omega\left(A \cap B^C\right). \tag{3.5}$$

Since $\omega \in A_\infty$, we have

$$\omega\left(A \cap B^C\right) \leq C_0 \left[\frac{\left|A \cap B^C\right|}{|D|}\right]^\nu \omega(D), \tag{3.6}$$

where $(C_0, \nu) = [\omega]_{A_\infty}$.

Using (3.3) and (3.6) in (3.5), we deduce that

$$\omega(A) \leq \omega(B) + C_0 (C\varepsilon)^\nu \omega(D). \tag{3.7}$$

Applying the inequality $(a+b)^r \leq 2^r\left(a^r + b^r\right)$ with $a, b, r \in \mathbb{R}^+$, it follows from (3.7) that

$$\omega(A)^{s/t} \leq 2^{s/t} \omega(B)^{s/t} + 2^{s/t} C_0^{s/t} (C\varepsilon)^{\nu s/t} \omega(D)^{s/t}. \tag{3.8}$$

We may rewrite the definition of the norm in weighted Lorentz space and change variables from λ to $\varepsilon^{-a}\lambda$ to get

$$\begin{aligned}\left\|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)\right\|^{s}_{L^{t,s}_{\omega}(\Omega)} &= t\int_0^{\infty}\lambda^{s}\omega\left(\left\{x\in\Omega:\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)>\lambda\right\}\right)^{s/t}\frac{d\lambda}{\lambda}\\ &= t\varepsilon^{-as}\int_0^{\infty}\lambda^{s}\omega\left(\left\{x\in\Omega:\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)>\varepsilon^{-a}\lambda\right\}\right)^{s/t}\frac{d\lambda}{\lambda}.\end{aligned} \tag{3.9}$$

Combining estimates (3.8) and (3.9), it follows that

$$\begin{aligned}\left\|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)\right\|^{s}_{L^{t,s}_{\omega}(\Omega)} &\leq 2^{s/t}\varepsilon^{-as}t\int_0^{\infty}\lambda^{s}\omega\left(\left\{x\in\Omega:\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\mathbf{F}\right)\right)>\varepsilon^{b}\lambda\right\}\right)^{s/t}\frac{d\lambda}{\lambda}\\ &+2^{s/t}C_0^{s/t}\left(C\varepsilon\right)^{\nu s/t}\varepsilon^{-as}t\int_0^{\infty}\lambda^{s}\omega\left(\left\{x\in\Omega:\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)>\lambda\right\}\right)^{s/t}\frac{d\lambda}{\lambda},\end{aligned}$$

which means

$$\left\|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)\right\|^{s}_{L^{t,s}_{\omega}(\Omega)} \leq 2^{s/t}\varepsilon^{-as-bs}\left\|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\mathbf{F}\right)\right)\right\|^{s}_{L^{t,s}_{\omega}(\Omega)}+2^{s/t}C_0^{s/t}C^{\nu s/t}\varepsilon^{\nu s/t-as}\left\|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)\right\|^{s}_{L^{t,s}_{\omega}(\Omega)}. \tag{3.10}$$

For $t\in\left(0,\nu a^{-1}\right)$, we may conclude (3.1) by taking $\varepsilon\in\left(0,\varepsilon_0\right)$ in (3.10) such that

$$2^{s/t}C_0^{s/t}C^{\nu s/t}\varepsilon^{\nu s/t-as}\leq\frac{1}{2}.$$

When $s=\infty$, we may rephrase the definition of the norm in weighted Lorentz space and change variables from λ to $\varepsilon^{-a}\lambda$ to get

$$\begin{aligned}\left\|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)\right\|_{L^{t,\infty}_{\omega}(\Omega)} &= \sup_{\lambda>0}\lambda\omega\left(\left\{x\in\Omega:\left|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)\right|>\lambda\right\}\right)^{1/t}\\ &= \sup_{\lambda>0}\lambda\varepsilon^{-a}\omega\left(\left\{x\in\Omega:\left|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)\right|>\lambda\varepsilon^{-a}\right\}\right)^{1/t}.\end{aligned} \tag{3.11}$$

Using the inequality $(a+b)^r\leq 2^r\left(a^r+b^r\right)$ with $a,b,r\in\mathbb{R}^+$, it follows from (3.7) that

$$\omega(A)^{1/t}\leq 2^{1/t}\omega(B)^{1/t}+2^{1/t}C_0^{1/t}\left(C\varepsilon\right)^{\nu/t}\omega(D)^{1/t}. \tag{3.12}$$

Combining estimates (3.11) and (3.12), it follows that

$$\begin{aligned}\left\|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)\right\|_{L^{t,\infty}_{\omega}(\Omega)} &\leq 2^{1/t}\varepsilon^{-a-b}\sup_{\lambda>0}\lambda\varepsilon^{b}\omega\left(\left\{x\in\Omega:\left|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\mathbf{F}\right)\right)\right|>\lambda\varepsilon^{b}\right\}\right)^{1/t}\\ &+2^{1/t}C_0^{1/t}\left(C\varepsilon\right)^{\nu/t}\varepsilon^{-a}\sup_{\lambda>0}\lambda\omega\left(\left\{x\in\Omega:\left|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)\right|>\lambda\right\}\right)^{1/t},\end{aligned}$$

which means

$$\left\|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)\right\|_{L^{t,\infty}_{\omega}(\Omega)}\leq 2^{1/t}\varepsilon^{-a-b}\left\|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\mathbf{F}\right)\right)\right\|_{L^{t,\infty}_{\omega}(\Omega)}+2^{1/t}C_0^{1/t}C^{\nu/t}\varepsilon^{\nu/t-a}\left\|\mathbf{M}_{\beta}\left(\mathcal{H}\left(x,\nabla u\right)\right)\right\|_{L^{t,\infty}_{\omega}(\Omega)}. \tag{3.13}$$

For $t\in\left(0,\nu a^{-1}\right)$, we may conclude (3.1) by taking $\varepsilon\in\left(0,\varepsilon_0\right)$ in (3.13) such that

$$2^{1/t}C_0^{1/t}C^{\nu/t}\varepsilon^{\nu/t-a}\leq\frac{1}{2}.$$

Next, we prove (3.2). Since $K(2z) \le cK(z)$ for all $z \ge 0$, it follows from Lemma 2.2 that when ε is small enough, there exist constants $I_1 > 0$ and $p_1 > 1$ such that for any $z > 0$, there holds

$$K\left(\varepsilon^{-a-b}z\right) \le I_1\varepsilon^{-ap_1-bp_1}K(z);\quad K\left(\varepsilon^{-a}z\right) \le I_1\varepsilon^{-ap_1}K(z). \tag{3.14}$$

Since p_1 only depends on function K in (3.14), we can choose a such that

$$0 < a < \min\left\{1-\frac{\beta}{n}, \frac{\nu}{p_1}\right\}. \tag{3.15}$$

For all $\lambda > 0$, by $\omega \in A_\infty$ and controlled condition (H), it is easily seen that

$$\begin{aligned}\omega\left(\left\{x\in\Omega : \mathbf{M}_\beta\left(\mathcal{H}\left(x,\nabla u\right)\right) > \varepsilon^{-a}\lambda\right\}\right) &\le \omega\left(\left\{x\in\Omega :\ \mathbf{M}_\beta\left(\mathcal{H}\left(x,\mathbf{F}\right)\right) > \varepsilon^{b}\lambda\right\}\right)\\ &\quad + C_0(C\varepsilon)^\nu\omega\left(\left\{x\in\Omega : \mathbf{M}_\beta\left(\mathcal{H}\left(x,\nabla u\right)\right) > \lambda\right\}\right).\end{aligned} \tag{3.16}$$

For all $z > 0$, let us apply (3.16) by $\lambda = \varepsilon^a K^{-1}\left(z\right)$, one gets

$$\begin{aligned}\omega\left(\left\{x\in\Omega : \mathbf{M}_\beta\left(\mathcal{H}\left(x,\nabla u\right)\right) > K^{-1}(z)\right\}\right) &\le \omega\left(\left\{x\in\Omega :\ \mathbf{M}_\beta\left(\mathcal{H}\left(x,\mathbf{F}\right)\right) > \varepsilon^{a+b}K^{-1}(z)\right\}\right)\\ &\quad + C_0(C\varepsilon)^\nu\omega\left(\left\{x\in\Omega : \mathbf{M}_\beta\left(\mathcal{H}\left(x,\nabla u\right)\right) > \varepsilon^{a}K^{-1}(z)\right\}\right),\end{aligned}$$

which guarantees that

$$\begin{aligned}\omega\left(\left\{x\in\Omega : K\left(\mathbf{M}_\beta\left(\mathcal{H}\left(x,\nabla u\right)\right)\right) > z\right\}\right) &\le \omega\left(\left\{x\in\Omega :\ K\left(\varepsilon^{-a-b}\mathbf{M}_\beta\left(\mathcal{H}\left(x,\mathbf{F}\right)\right)\right) > z\right\}\right)\\ &\quad + C_0(C\varepsilon)^\nu\omega\left(\left\{x\in\Omega : K\left(\varepsilon^{-a}\mathbf{M}_\beta\left(\mathcal{H}\left(x,\nabla u\right)\right)\right) > z\right\}\right),\end{aligned} \tag{3.17}$$

since K is a strictly increasing function. Thanks to (3.14), we can deduce from (3.17) that

$$\begin{aligned}\omega\left(\left\{x\in\Omega : K\left(\mathbf{M}_\beta\left(\mathcal{H}\left(x,\nabla u\right)\right)\right) > z\right\}\right) &\le \omega\left(\left\{x\in\Omega :\ \alpha_1 K\left(\mathbf{M}_\beta\left(\mathcal{H}\left(x,\mathbf{F}\right)\right)\right) > z\right\}\right)\\ &\quad + C_0(C\varepsilon)^\nu\omega\left(\left\{x\in\Omega : \alpha_2 K\left(\mathbf{M}_\beta\left(\mathcal{H}\left(x,\nabla u\right)\right)\right) > z\right\}\right),\end{aligned} \tag{3.18}$$

where $\alpha_1 = I_1\varepsilon^{-ap_1-bp_1}$ and $\alpha_2 = I_1\varepsilon^{-ap_1}$.

Integrating two sides of (3.18) over the range $\left[0,\infty\right)$ and then changing the variable on the right hand side, we have

$$\begin{aligned}\int_0^\infty \omega\left(\left\{x\in\Omega : K\left(\mathbf{M}_\beta\left(\mathcal{H}\left(x,\nabla u\right)\right)\right) > z\right\}\right)dz &\le \alpha_1\int_0^\infty \omega\left(\left\{x\in\Omega :\ K\left(\mathbf{M}_\beta\left(\mathcal{H}\left(x,\mathbf{F}\right)\right)\right) > z\right\}\right)dz\\ &\quad + C_0(C\varepsilon)^\nu\alpha_2\int_0^\infty \omega\left(\left\{x\in\Omega : K\left(\mathbf{M}_\beta\left(\mathcal{H}\left(x,\nabla u\right)\right)\right) > z\right\}\right)dz.\end{aligned} \tag{3.19}$$

One notes that

$$\int_\Omega K\left(\mathbf{M}_\beta\left(\mathcal{H}(x,\nabla u)\right)\right)\omega(x)dx = \int_0^\infty \omega\left(\left\{x\in\Omega : K\left(\mathbf{M}_\beta\left(\mathcal{H}(x,\nabla u)\right)\right) > z\right\}\right)dz,$$

so we may choose $\varepsilon \in (0,\varepsilon_0)$ in (3.19) such that $C_0(C\varepsilon)^\nu \alpha_2 = C_0 C^\nu I_1 \varepsilon^{\nu - ap_1} \le \frac{1}{2}$ to obtain (3.2).

Theorem 3.2 (Pointwise estimate for the Riesz potential) Let $\beta \in [0,n)$ and let (P) be a problem as defined in Section 1 under conditions (A1)-(A3) on an open bounded domain Ω. Assume $\partial\Omega$ belongs to C^{1,κ^+} class for some $\kappa^+ \in [\kappa, 1]$. Suppose that $u \in W_0^{1,1}(\Omega)$ is a distributional solution to (P) with a given data satisfying $\mathcal{H}(x,\nabla u),\ \mathcal{H}(x,\mathbf{F}) \in L^1(\Omega)$.
For any $0<\alpha<n$, there exists a positive constant C^* such that

$$\mathbf{I}_\alpha\left[\mathbf{M}_\beta\left(\mathcal{H}(x,\nabla u)\right)\right] \le C^* \mathbf{I}_\alpha\left[\mathbf{M}_\beta\left(\mathcal{H}(x,\mathbf{F})\right)\right], \tag{3.20}$$

holds for almost everywhere $x \in \mathbb{R}^n$.

Proof. Applying Theorem 3.1 with $K(x)=x$ and $\omega \in A_\infty$, there exists a constant C^* only depending on $data, \Omega, \beta, [\omega]_{A_\infty}$ such that

$$\int_\Omega \mathbf{M}_\beta\left(\mathcal{H}(x,\nabla u)\right)\omega(x)dx \le C^* \int_\Omega \mathbf{M}_\beta\left(\mathcal{H}(x,\mathbf{F})\right)\omega(x)dx. \tag{3.21}$$

holds for any $z \in \mathbb{R}^n$ and $\varepsilon > 0$ small enough. We may choose $h = \chi_{B_\varepsilon(z)} \in L^1_{loc}\left(\mathbb{R}^n;\mathbb{R}^+\right)$ and let $\omega = I_\alpha h$. We will verify that $\omega \in A_\infty$ by showing $\omega \in A_1$, since $A_1 \subset A_\infty$.
Indeed, it is not difficult to prove that for $\omega_0(x) = |x|^{1-n}$ there exists a constant $L>0$ such that

$$\mathbf{M}(\omega_0)(x) \le L\omega_0(x), \tag{3.22}$$

for all $x \in \mathbb{R}^n$. Using Fubini's theorem, it implies that $I_\alpha h$ satisfies the following inequalities for all $h \in L^1_{loc}\left(\mathbb{R}^n;\mathbb{R}^+\right)$ and $x \in \mathbb{R}^n$

$$\mathbf{M}(I_\alpha h)(x) \le L I_\alpha h(x), \tag{3.23}$$

which demonstrates that $I_\alpha h$ belongs to A_1.
Now, we may use the chosen function $\omega = I_\alpha\left[\chi_{B_\varepsilon(z)}\right]$ in (3.21), which gives:

$$\int_{\mathbb{R}^n} \mathbf{M}_\beta\left(\mathcal{H}(x,\nabla u)\right) \int_{\mathbb{R}^n} \frac{\chi_{B_\varepsilon(z)}(y)}{|y-x|^{n-\alpha}} dydx \le C^* \int_{\mathbb{R}^n} \mathbf{M}_\beta\left(\mathcal{H}(x,\mathbf{F})\right) \int_{\mathbb{R}^n} \frac{\chi_{B_\varepsilon(z)}(y)}{|y-x|^{n-\alpha}} dydx.$$

Note that now C^* only depends on $data, \Omega, \beta$. Thanks to Fubini's theorem again, it leads to the following estimate

$$\int_{\mathbb{R}^n} \chi_{B_\varepsilon(z)}(y) \int_{\mathbb{R}^n} \frac{\mathbf{M}_\beta\left(\mathcal{H}(x,\nabla u)\right)}{|y-x|^{n-\alpha}} dxdy \le C^* \int_{\mathbb{R}^n} \chi_{B_\varepsilon(z)}(y) \int_{\mathbb{R}^n} \frac{\mathbf{M}_\beta\left(\mathcal{H}(x,\mathbf{F})\right)}{|y-x|^{n-\alpha}} dxdy,$$

which can be rewritten as

$$\int_{B_\varepsilon(z)} I_\alpha\left[\mathbf{M}_\beta\left(\mathcal{H}(.,\nabla u)\right)\right](y)\, dy \le C^* \int_{B_\varepsilon(z)} I_\alpha\left[\mathbf{M}_\beta\left(\mathcal{H}(.,\mathbf{F})\right)\right](y)\, dy. \tag{3.24}$$

Letting ε tend to 0 in (3.24) and applying Lebesgue differentiable theorem, we obtain that (3.20) holds almost everywhere for $z \in \mathbb{R}^n$. The proof is complete.

❖ ***Conflict of Interest:*** *Author have no conflict of interest to declare.*

REFERENCES

Byun, S. S., & Oh, J. (2017). Global gradient estimates for non-uniformly elliptic equations. *Calc. Var. Partial Differ. Equ.*, *56*(2).

Colombo, M., & Mingione, G. (2016). Calderón-Zygmund estimates and non-uniformly elliptic operators. *J. Funct. Anal.*, *270*(4), 1416-1478.

Filippis, C. De, & Mingione, G. (2019). A borderline case of Calderón-Zygmund estimates for non-uniformly elliptic problems. *St. Petersburg Math. J.*, *31*(3), 82-115.

Harjulehto, P., & Hasto, P. (2019). *Generalized Orlicz spaces*. Lecture notes in Mathematics, Springer International Publishing.

Nguyen, T. N., & Tran, M. P. (2021). Level-set inequalities on fractional maximal distribution functions and applications to regularity theory. *Journal of functional analysis*, 280, 108797.

Nguyen, T. N., & Tran, M. P. (2021). Lorentz improving estimates for the p-Laplace equations with mixed data. *Nonlinear Analysis 200*, 111960.

Tran, M. P., & Nguyen, T. N. (2019). Generalized good-λ techniques and applications to weighted Lorentz regularity for quasilinear elliptic equations. *Comptes Rendus Mathematique*, *357*(8), 664-670.

Tran, M. P., & Nguyen, T. N. (2021). Global Lorentz estimates for non-uniformly nonlinear elliptic equations via fractional maximal operators. *Journal of mathematical analysis and applications*, 501, 124084.

ĐÁNH GIÁ GRADIENT TOÀN CỤC VÀ TỪNG ĐIỂM CHO NGHIỆM CỦA BÀI TOÁN PHA KÉP TRONG KHÔNG GIAN ORLICZ

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TÓM TẮT

Bài báo này có hai mục đích. Thứ nhất, chúng tôi đưa ra một ước lượng toàn cục dạng Calderón-Zygmund cho nghiệm của bài toán pha kép trong không gian Orlicz sử dụng toán tử cực đại cấp phân số. Phương pháp chúng tôi sử dụng trong nghiên cứu này được dựa trên kĩ thuật good-λ tổng quát được phát triển bởi Tran, và Nguyen, 2019, trong đó các kết quả về tính chính quy nghiệm được bảo toàn qua toán tử cực đại cấp phân số. Toán tử này được biết đến rộng rãi qua vai trò của nó trong việc ước lượng sự dao động của các hàm số, và có một mối liên hệ gần gũi giữa nó và thế vị Riesz. Trong kết quả thứ hai, chúng tôi trình bày ước lượng từng điểm cho thế vị Riesz như là một hệ quả của kết quả thứ nhất.

Từ khóa: bài toán pha kép; không gian Orlicz; đánh giá gradient; thế vị Riesz; toán tử cực đại cấp phân số