Sufficient optimality conditions for the optimal control problem of 2D *g*-Navier-Stokes equations

Điều kiện đủ tối ưu cho bài toán điều khiển tối ưu của hệ phương trình *g*-Navier-Stokes hai chiều

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Abstract

In this paper, we establish second-order sufficient optimality conditions for optimal control problem of 2D *g*-Navier-Stokes equations.

Keywords: *2D g-Navier-Stokes equations; weak solution; sufficient optimality conditions.*

Tóm tắt

Trong bài báo này, chúng tôi thiết lập điều kiện đủ tối ưu bậc hai cho bài toán điều khiển tối ưu đối với hệ phương trình *g*-Navier-Stokes hai chiều.

Từ khóa: *Hệ phương trình g-Navier-Stokes equations; nghiệm yếu; điều kiện đủ tối ưu.*

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^2$ with sufficiently smooth boundary Γ. In this paper we consider the following two-dimensional *g*-Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \nu\Delta y + (y.\Delta)y + \nabla p = f \text{ in } (0,T)\times\Omega, \\ \nabla\cdot(gy) = 0 \text{ in } (0,T)\times\Omega, \\ y(0,x) = y_0,\ x\in\Omega. \end{cases} \quad (1)$$

Where:

$y = y(x, t) = (y_1, y_2)$ is the unknown velocity vecto;

$p = p(x, t)$ is the unknown pressure;

$\nu > 0$ is the kinematic viscosity coefficient;

y_0 is the initial velocity.

The *g*-Navier-Stokes equations is a variation of the standard Navier-Stokes equations. More precisely, when $g \equiv$ const we get the usual Navier-Stokes equations. The 2D *g*-Navier-Stokes equations arise in a natural way when we study the standard 3D problem in thin domains. We refer the reader to [7] for a derivation of the 2D *g*-Navier-Stokes equations from the 3D Navier-Stokes equations and a relationship between them. As mentioned in [7], good properties of the 2D *g*-Navier-Stokes equations can lead to an initiate the study of the Navier-Stokes equations on the thin three dimensional domain $\Omega g = \Omega \times (0, g)$. In the last few years, the existence and asymptotic behavior of weak solutions to 2D *g*-Navier-Stokes equations have been studied extensively in (see e.g. [1, 5, 7]). In a recent work [2], we proved the existence and numerical approximation of strong solutions to the two-dimensional *g*-Navier-Stokes. The long-time behavior of the strong solutions was studied very recently in [3] in the autonomous case in terms of existence of a global attractor, and existence and stability of a unique stationary solution.

In this paper, we consider an optimal control with a quadratic objective functional for 2D *g*-Navier-Stokes equations. To do this, we assume that the function g satisfies the following assumption:

$(G)\ g \in W^{1,\infty}(\Omega)$

Such that:

$0 < m_0 \le g(x) \le M_0$ for all $x = (x_1, x_2) \in \Omega$, and $|\nabla g|_\infty < m_0\lambda_1^{1/2}$.

Where:

$\lambda_1 > 0$ is the first eigenvalue of the *g*-Stokes operator in Ω (i.e. the operator A defined in Section 2 below).

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2. Dr. Dao Trong Quyet

The rest of the paper is organized as follows. In Section 2, for convenience of the reader, we recall some auxiliary results on the 2D *g*-Navier-Stokes equations, which will be used later. Next section, we show the optimal control problem, and the main results is shown in the last section.

2. PRELIMINARY RESULTS

To consider equation (1), we denote by $Q = \Omega \times (0, T)$ the space-time cylinder.

Here, $T < \infty$ is a given final time. Futher, we set $\Sigma := \Gamma \times (0,T)$.

Let $L^2(\Omega,g) = (L^2(\Omega))^2$ and $L_0^1(\Omega,g) = (L_0^1(\Omega))^2$ be endowed, respectively, with the inner products.

$$(u,\upsilon)_g = \int_\Omega u.\upsilon g dx,\ u,\upsilon \in L^2(\Omega,g),$$

And

$$((u,\upsilon))_g = \int_\Omega \sum_{j=1}^2 \nabla u_j \nabla \upsilon_j g dx,\ u = (u_1,u_2),$$

$$\upsilon = (\upsilon_1,\upsilon_2) \in H_0^1(\Omega,g),$$

And norms $|u|^2 = (u,u)_g, \|u\|^2 = ((u,u))_g$. Thanks to assumption ($G$), the norms $|\cdot|$ and $\|\cdot\|$ are equivalent to the usual ones in $(L^2(\Omega))^2$ and in $(H_0^1(\Omega))^2$.

Let

$$v = \left\{u \in \left(C_0^\infty(\Omega)\right)^2 : \nabla.(gu) = 0\right\}.$$

Denote by H_g the closure of v in $(L^2(\Omega))^2$ and by V_g the closure of v in $(H_0^1(\Omega))^2$. It follows that $V_g \subset H_g \equiv H_g' \subset V_g'$, where the injections are dense and continuous. We will use $\|\cdot\|_*$ for the norm in V_g' and $\langle .,. \rangle$ for duality pairing between V_g and V_g'. To deal with the time derivative in the state equation, we introduce the common spaces of functions y whose time derivatives y_t exists as abstact functions.

$$W^\alpha(0,T;V_g) := \left\{y \in L^2(0,T;V_g) : y_t \in L^\alpha(0,T;V_g')\right\},$$

$$W(0,T) := W^\alpha(0,T;V_g).$$

Where $1 \le \alpha \le 2$. Endowed with the norm

$$\|y\|_{W^\alpha} := \|y\|_{W^\alpha(0,T;V_g)} = \|y\|_{L^2(V_g)} + \|y_t\|_{L^\alpha(V_g')},$$

$$\|y\|_W := \|y\|_{W^2}.$$

These spaces are Banach spaces.

Set $A : V_g \to V_g'$ by $\langle Au,\upsilon \rangle = ((u,\upsilon))_g$,

Denote $D(A) = \{u \in V_g : Au \in H_g\}$

Then $D(A) = H^2(\Omega,g) \cap V_g$

And $Au = -P_g \Delta u, \forall u \in D(A)$

Where P_g is the ortho-projector from $L^2(\Omega,g)$ onto H_g.

Set $B : V_g \times V_g \to V_g'$ by $\langle B(u,\upsilon),w \rangle = b(u,\upsilon,w)$ where

$$b(u,\upsilon,w) = \sum_{j,k=1}^2 \int_\Omega u_j \frac{\partial \upsilon_k}{\partial x_j} w_k g dx$$

Whenever the integrals make sence. It is easy to check that if $u,\upsilon,w \in V_g$, then

$$b(u,\upsilon,w) = -b(u,w,\upsilon) \tag{2}$$

Hence

$$b(u,\upsilon,\upsilon) = 0, \forall u,\upsilon \in V_g.$$

Let $u \in L^2(\tau,T,V_g)$, then the function C_u defined by

$(Cu(t),v)_g = \left(\left(\frac{\nabla g}{g}.\nabla\right)u,\upsilon\right)_g = b\left(\frac{\nabla g}{g},u,\upsilon\right), \forall \upsilon \in V_g$, belongs to $L^2(\tau,T,H_g)$, and hence also belongs to $L^2(\tau,T,V_g')$.

$$-\frac{1}{g}(\nabla.g\nabla)u = -\Delta u - \left(\frac{\nabla g}{g}.\nabla\right)u$$

We have

$$(-\Delta u,\upsilon)_g = ((u,\upsilon))_g + \left(\left(\frac{\nabla g}{g}.\nabla\right)u,\upsilon\right)_g$$

$$= (Au,\upsilon)_g + \left(\left(\frac{\nabla g}{g}.\nabla\right)u,\upsilon\right)_g, \forall u,\upsilon \in V_g.$$

Definition 2.1 (Weak solution). Let $f \in L^2(0,T;V_g')$ and $y_0 \in H_g$ be given, a weak solution of problem (1.1) is a function $y \in L^2(0,T;V_g)$ with $y_t \in L^2(0,T;V_g')$, such that

$$\begin{cases} y_t + \nu Ay + \nu Cy + B(y) = f \ \text{ in } L^2(0,T;V_g'), \\ y(0) = y_0 \ \text{ in } H_g. \end{cases} \tag{3}$$

Theorem 2.1 ([1]). *(Existence and uniqueness of solutions)*

Let Ω be a bounded and locally Lipschitz domain in $\mathbb{R}$. Then for every $f \in L^2(0,T;V_g')$ and $y_0 \in H_g$, the equation (3) has a unique solution $y \in W(0,T)$.

2.1. Linearized equations. We will need in the following some results about linearied equations. Given a state $\bar{y} \in W(0,T)$, we consider the system

$$\begin{cases} y_t + \nu Ay + \nu Cy + B'(\bar{y})y = f \ \text{ in } L^2(0,T;V_g'), \\ y(0) = y_0 \ \text{ in } H_g. \end{cases} \tag{4}$$

Here, $B'(\bar{y})y$ denotes the Frechet derivative of B with respect to the state $\bar{y}$. It is itself a functional of $L^2(0,T;V_g')$, which for $\upsilon \in L^2(0,T;V_g)$ is given by

$$\begin{aligned}&\left(B'(\bar{y})y,v\right)\\ &= \int_0^T \left(b(\bar{y}(t),y(t),v(t)) + b(y(t),\bar{y}(t),v(t))\right)dt.\end{aligned} \tag{5}$$

Lemma 2.1. *([4]) The operator* $B : W(0,T) \to L^2(0,T;V_g')$ is twice Frechet differentiable. All derivatives of third or higher order vanish. The first derivative is given by (2). It can be estimated as.

$$\|B'(\bar{y})y\|_{L^2(V_g')} \le c\|\bar{y}\|w\|y\|_W \tag{6}$$

As for quadratic functions, the second derivative is independent of $\bar{y}$:

$$\left(B''(\bar{y})[y_1,y_2],\upsilon\right) = \int_0^T \left(b(y_1(t),y_2(t),\upsilon(t)) + b(y_2(t),y_1(t),\upsilon(t))\right)dt \quad (7)$$

The adjoint of $B'(\bar{y})$, called $B'(\bar{y})^*$, is a linear and continuous operator from $L^2(0,T;V_g)$ to $W^*(0,T)$. It can be written as

$$\left(B'(\bar{y})^*\lambda,w\right) = \int_0^T \left(b(\bar{y}(t),w(t),\lambda(t)) + b(w(t),\bar{y}(t),\lambda(t))\right)dt$$

Theorem 2.2 ([4]). *Let $f \in L^2(0,T;V_g')$, $y_o \in H_g$ and $\bar{y} \in W(0,T)$ be given. Then the equation (4) has a unique weak solution $y \in W(0,T)$.*

2.2. The control-to-state mapping

We will study the mapping: Right-hand side $\mapsto$ solution, the so-called control to state mapping. Let $u \in L^2(Q)^2$ denote the control, then we will use u as f in (1).

Definition 2.2. (*Solution mapping*)

Consider the system (1). The mapping $u \mapsto y$, where y is the weak solution of (1) with the control right-hand side u and fixed initial value y_0, is denoted by S, i.e. $y = S(u)$.

Lemma 2.2 ([4]). *The control-to-state mapping is Frechet differentiable as mapping from $L^2(0,T;V_g')$ to $W(0, T)$. The derivative $\bar{u} \in L^2(0,T;V_g')$ in direction $h \in L^2(0,T;V_g')$ is given by $S'(\bar{u})h = y$, where y is the weak solution of $S'(\bar{u})$*

$$\begin{cases} y_t + \nu Ay + \nu Cy + B(\bar{y})y = h & \text{in } L^2(0,T;V_g'), \\ y(0) = y_0 & \text{in } H_g. \end{cases} \quad (8)$$

With $\bar{y} = S(\bar{u})$.

In order to establish first-order optimality conditions, we will need the adjoint operator of $S'(\bar{u})$ denote by $S'(\bar{u})^*$. By Lemma 2.2, we can regard $S'(\bar{u})$ as linear operator from $L^2(0,T;V_g')$ to W_0, where W_0 is defined as a closed linear subspace of $W(0, T)$ by

$$W_0 := \{y \in W(0,T) : y(0) = 0\}. \quad (9)$$

Hence, the adjoint will be a mapping from W_0^* to $L^2(0,T;V_g)$.

Lemma 2.3 ([4]). *Let be $\bar{u} \in L^2(Q)^2$. Then the operator $S'(\bar{u})^*$ is linear and continuous from W_0^* to $L^2(0,T;V_g')$. Its action is defined as follows. Take z in W_0^*, then $\lambda = S'(\bar{u})^* z$ holds if and only if*

$$\left\langle w_t + \nu Aw + \nu Cw + B'(\bar{y})w, \lambda \right\rangle_{L^2(V_g'),L^2(V_g)} = \langle z, w\rangle_{W_0^*,W_0} \quad (10)$$

For all $w \in W_0$.

Lemma 2.4 ([4]). *Let be $\bar{u} \in L^2(Q)^2$ given. Suppose the right-hand side z of (10) is in the form $z = z_1 + z_2$ with functionals $z_1 \in L^{4/3}(0,T;V_g') \cap W_0^*$ and $z_2 \in W_0^*$ defined by $z_2(w) = (z_T, w(T)), z_T \in H_g$. Then $\lambda = S'(\bar{u})^* z$ is the weak solution of*

$$\begin{cases} -\lambda_t + \nu A\lambda + \nu C\lambda + B'(\bar{y})^*\lambda = z_1 & \text{in } L^{4/3}(0,T;V_g'), \\ \lambda(T) = z_T. \end{cases} \quad (11)$$

Futhermore, it holds $\lambda \in W^{4/3}(0,T)$.

3. THE OPTIMAL CONTROL PROBLEM

We are considering optimal control of the instationary *g*-Navier-Stokes equations. The minimization of the following quadratic objective functional serves as model problem:

$$J(y,u) = \frac{\alpha_T}{2}\int_\Omega |y(x,T) - y_T(x)|^2 dx + \frac{\alpha_Q}{2}\int_\Omega |y(x,t) - y_Q(x,t)|^2 dxdt + \frac{\gamma}{2}\int_\Omega |u(x,t)|^2 dxdt. \quad (12)$$

They are weighted by the coefficients α_T, α_Q, α_R and γ. The free variables-state y and control u - have to fulfill the instationary *g*-Navier-Stokes equations

$$\begin{cases} y_t - \nu\Delta y + (y\cdot\Delta)y + \nabla p = u & \text{in } Q, \\ \nabla\cdot(gy) = 0 & \text{in } Q, \\ y = 0 & \text{in } \Sigma, \\ y(0,x) = y_0 & \text{in } \Omega. \end{cases} \quad (13)$$

The control has to satisfy inequality constraints:

$u_{a,i}(x,t) \le u_i(x,t) \le u_{b,i}(x,t)$ a.e. on Q, $i = 1,2$.

3.1. Setting of the problem

Let us specify the problem setting. Unless other conditions are imposed, we assume that the ingredients of the optimal problem satisfy the following:

i) The domain Ω is supposed to be an open bounded subset of $\mathbb{R}^2$ with Lipschitz boundary Γ. We denote the time-space cylinder by $Q = \Omega \times (0, T)$ its boundary by $\Sigma = \Gamma \times (0,T)$.

ii) The initial value y_0 is a given function in H_g. The desirred states have to satisfy $y_T \in H_g$ and $y_Q \in L^2(Q)^2$.

iii) The parameter ν is positive real number. The coefficients α_T, α_Q are non-negative real number, where at least one of them is positive to get a non-trivial objective functional. The regularization parameter γ, which measures the cost of the control, is also a positive number.

iv) The control constraints $u_a, u_b \in L^2(Q)^2$ have to satisfy $u_{a,i}(x,t) \le u_{b,i}(x,t)$ a.e. on Q for $i = 1,2$.

We define the set of admissible controls U_{ad} by

$$U_{ad}=\begin{Bmatrix} u\in L^2(Q)^2 : u_{a,i}(x,t)\le u_i(x,t)\le u_{b,i}(x,t) \\ \text{a.e. on } Q, i=1,2 \end{Bmatrix}.$$

U_{ad} is non-empty, convex and closed in $L^2(Q)^2$.

So we end up with the optimization problem in function space $\min J(y,u)$.

Subject to the state equation

$$\begin{cases} y_t+\nu Ay+\nu Cy+B(y)=u \text{ in } L^2(0,T;V_g'), \\ y(0)=y_0. \end{cases} \tag{14}$$

And the control constraint

$$u\in U_{ad}. \tag{15}$$

3.2. Existence of solutions

We call a couple (y, u) of state and control admissible if it satisfies the constraints (14) - (15) of the optimal control problem. We will denote in the sequel pairs of control and state by u, e.g. $\upsilon=(y,u),\bar{\upsilon}=(\bar{y},\bar{u})$ and so on.

At first, we recall the optimal control problem has a solution.

Theorem 3.1 ([13]). *The optimal control problem admits a-globally optimal-solution $\bar{u}\in U_{ad}$ with associated state $\bar{y}\in W(0,T)$.*

3.3. Lagrange functional

We will define the Lagrange functional

$L: W(0,T)\times L^2(Q)^2\times L^2(0,T;V_g)\mapsto \mathbb{R}$ for the optimal control problem as follows:

$$L(y,u,\lambda)=J(y,u) -\langle y_t+\nu Ay+\nu Cy+B(y)-u,\lambda\rangle_{L^2(V_g'),L^2(V_g)}.$$

The Lagrange function L is, for given $\lambda\in L^2(0,T;V_g)$, twice Frechet-differentiable with respect to $(y,u)\in W(0,T)\times L^2(Q)^2$. The first-order derivatives of L with respect to y and u in direction $w\in W(0,T)$ and $h\in L^2(Q)^2$ respectively are

$$L_y(y,u,\lambda)w =\alpha_T(y(T)-y_T,w(T))_{H_g}+\alpha_Q(y-y_Q,w)_Q -\langle w_t+\nu Aw+\nu Cw+B'(y)w,\lambda\rangle_{L^2(V_g'),L^2(V_g)}.$$

$$L_u(y,u,\lambda)h=\gamma(u,h)_Q+(h,\lambda)_Q.$$

The second-order derivative of L with respect to $v=(y,u)$ in directions

$(w_1,h_1),(w_2,h_2)\in W(0,T)\times L^2(Q)^2$

Is

$$L_{\upsilon\upsilon}(y,u,\lambda)[(w_1,h_1),(w_2,h_2)] =L_{yy}(y,u,\lambda)[w_1,w_2]+L_{uu}(y,u,\lambda)(h_1,h_2)$$

With

$$L_{yy}(y,u,\lambda)[w_1,w_2]=\alpha_T(w_1(T),w_2(T))_{H_g} +\alpha_Q(w_1,w_2)_Q-\langle B''(y)[w_1,w_2],\lambda\rangle_{L^2(V_g'),L^2(V_g)}$$

And

$$L_{uu}(y,u,\lambda)[h_1,h_2]=\gamma(h_1,h_2)_Q.$$

Theorem 3.2 ([4]). *Let $\lambda\in L^2(0,T;V_g)$ be given. Then the Lagrangian L is twice Frechet-differentiable with respect to $v=(y,u)$ from $W(0,T)\times L^2(Q)^2$ to $\mathbb{R}$. The second-order derivative at (y,u) fulfill, together with the Lagrange multiplier λ, the estimate.*

$$|L_{yy}(y,u,\lambda)[w_1,w_2]|\le c_L(1+\|\lambda\|)\|w_1\|\|w_2\| \tag{16}$$

For all with some constant $c_L>0$ that does not depend on $w_1,w_2\in W(0,T)$.

Remark 3.1. *The objective functional J is twice continuously di erentiable from $W(0,T)\times L^2(Q)^2$ to $\mathbb{R}$. The reduced objective $\phi(u):=J(S(u),u)$ continuously is also twice continuous di erentiable from $L^2(Q)^2$ to $\mathbb{R}$.*

Now, we recall first-order necessary optimality condition of the optimal control problem, it have been showed in [13].

Definition 3.1. *A control $\bar{u}\in U_{ad}$ is said to be locally optimal in $L^p(Q)^2$, if there exists a constant ρ > 0 such that $J(\bar{y},\bar{u})\le J(y,u)$ holds for all $u\in U_{ad}$ with $\|\bar{u}-u\|_p\le\rho$. Where $\bar{y}$ and y denote the states associated with $\bar{u}$ and u.*

Now, we will state the first-order optimality condition:

Theorem 3.3 ([13]). *Let $\bar{u}$ be locally optimal in $L^2(Q)^2$ with associated state $\bar{y}=S(\bar{u})$. Then there exists a unique Lagrange $\bar{\lambda}\in W^{4/3}(0,T;V_g)$, which is the weak solution of the adjoint equation.*

$$-\bar{\lambda}_t+\nu A\bar{\lambda}+\nu C\bar{\lambda}+B'(\bar{y})^*\bar{\lambda}=\alpha_Q(\bar{y}-y_Q)$$
$$\bar{\lambda}(T)=\alpha_T(\bar{y}(T)-y_T).$$

Moreover, the variational inequality

$$(\gamma\bar{u}+\bar{\lambda},u-\bar{u})_Q\ge 0\quad \forall u\in U_{ad}$$

is satisfied.

Theorem 3.4. *Under the conditions of Theorem 3.3, it is necessary for local optimality of $\bar{u}$ that there exists $\lambda\in W^{4/3}(0,T)$, such that*

$$L_y(\bar{y},\bar{u},\bar{\lambda})w=0\quad \forall w\in W_0,$$

$$L_y(\bar{y},\bar{u},\bar{\lambda})(u-\bar{u})\ge 0\quad \forall u\in U_{ad}$$

is fulfilled.

4. SECOND-ORDER SUFFICIENT OPTIMALITY CONDITIONS

The sufficient condition that we will present here requires coercivity of the second derivative of the Lagrangian only for a subspace of the space of all possible directions. Using strongly active constraints, we are able to shrink the subspace in which the coercivity must hold.

Now, let us specify the notations of strongly active sets.

Definition 4.1. *(Strong active sests). Let* $\varepsilon > 0$ *be given. Define sets* $Q_{\varepsilon,i} \subseteq Q = \Omega\times[0,T]$ *for* $i = \{1, 2\}$ *by*

$$Q_{\varepsilon,i} = \left\{(x,t)\in Q : \left|\gamma \bar{u}_i(x,t) + \bar{\lambda}_i(x,t) > \varepsilon\right|\right\}$$

For $u \in L^p(Q)^2$ and $1 \le p < \infty$, we define the L^p-norm with respect to the sets of strongly active control constraints.

$$\|u\|_{L^p(Q_\varepsilon)} := \left(\sum_{i=1}^{2} \|u_i\|^p_{L^p(Q_\varepsilon)}\right)^{1/p}.$$

Remark 4.1. *Note that the variational inequality (13) uniquely determines* $\bar{u}_i$ *on* $Q_{\varepsilon,i}$. *If* $\gamma\bar{u}_i(x,t)+\bar{\lambda}_i(x,t) \ge \varepsilon$ *then* $\bar{u}_i(x,t) = u_{a,i}(x,t)$ *must hold. On the other hand, it follows* $\bar{u}_i(x,t) = u_{b,i}(x,t)$ *if* $\gamma\bar{u}_i(x,t)+\bar{\lambda}_i(x,t) < -\varepsilon$ *is satisfied.*

In what follows we fix $\bar{\upsilon} := (\bar{y},\bar{u})$ to be an admissible reference pair. We suppose that $\bar{\upsilon}$ satisfies together with the adjoint state the first-order necessary optimality conditions, e.g. equations (1)-(13). Futhermore, we assume that the reference pair $\bar{\upsilon} := (\bar{y},\bar{u})$ satisfies for some $g \ge 3/4$ the following coercivity assumption on $L''(\bar{\upsilon},\bar{\lambda})$, in the sequel second-order su cient condition:

$$(SSC)\begin{cases} \text{There exist } \varepsilon > 0 \text{ and } \delta > 0 \text{ such that} \\ L_{\upsilon\upsilon}(\bar{\upsilon},\bar{\lambda})[z,h]^2 \ge \delta\|h\|_q^2, \\ \text{hold for all pairs } (z,h)\in W(0,T)\times L^2(Q)^2 \\ \text{with } h = u-\bar{u}, u\in U_{ad}, h_i = 0 \text{ on } Q_{\varepsilon,i} \text{ for } i = 1,2, \\ \text{and } z\in W(0,T) \text{ being the weak} \\ \text{solution of the linearied equation} \\ z_t + \nu Az + \nu Cz + B'(\bar{y})z = h \\ z(0) = 0. \end{cases}$$

The following theorem states the sufficiency of (SSC):

Theorem 4.1. *Let* $\bar{\upsilon} = (\bar{y},\bar{u})$ *be admissible for the optimal control problem and suppose that* $\bar{\upsilon}$ *fulfills the first order necessary optimality conditions with associated adjoint state. Assume further that (SSC) is satisfies at* $\bar{\upsilon}$ *with* $g \ge 3/4$. *Then* $\bar{u}$ *is locally optimal in* L^s. *Moreover, there exists* $\alpha > 0$ *and* $\rho > 0$ *such that.*

$$J(\upsilon) \ge J(\bar{\upsilon}) + \alpha\|u-\bar{u}\|_q^2$$

holds for admissible pairs $v = (y, u)$ *with* $\|u-\bar{u}\|_s \le \rho$, *where the exponent s is given by* $1 = 1/s + 1/q$.

Proof. Throughout the proof, c is used as a generic constant. Suppose that $\bar{\upsilon} = (\bar{y},\bar{u})$ fulfills the assumptions of the theorem. Let (y, u) be another admissible pair. We have

$$J(\bar{\upsilon}) = L(\bar{\upsilon},\bar{\lambda}) \text{ and } J(\upsilon) = L(\upsilon,\bar{\lambda}),$$

since $\bar{\upsilon}$ and υ are admissible. Taylor-expression of the Lagrange-function yields.

$$L(\upsilon,\bar{\lambda}) = L(\bar{\upsilon},\bar{\lambda}) + L_y(\bar{\upsilon},\bar{\lambda})(y-\bar{y})$$
$$+L_u(\bar{\upsilon},\bar{\lambda})(u-\bar{u}) + \frac{1}{2}L_{\upsilon\upsilon}(\bar{\upsilon},\bar{\lambda})[\upsilon-\bar{\upsilon},\upsilon-\bar{\upsilon}].$$

Notice that there is no remainder term due to the quadratic nature of all nonlin-earities. Moreover, the necessary condition is satisfied at $\bar{\upsilon}$ with adjoint state $\bar{\lambda}$. Therefore, the second term vanishes. The third term is nonnegative due to the variational inequality. However, we get even more by [4, Lemma 4.4],

$$L_u(\bar{\upsilon},\bar{\lambda})(u-\bar{u}) = \int_Q (\gamma\bar{u}+\bar{\lambda})(u-\bar{u})dxdt \ge \varepsilon\|u-\bar{u}\|_{L^1(Q_\varepsilon)}.$$

So we arrive at

$$J(\upsilon) = J(\bar{\upsilon}) + L_y(\bar{\upsilon},\bar{\lambda})(y-\bar{y})$$
$$+L_u(\bar{\upsilon},\bar{\lambda})(u-\bar{u}) + \frac{1}{2}L_{\upsilon\upsilon}(\bar{\upsilon},\bar{\lambda})[\upsilon-\bar{\upsilon}]^2$$
$$\ge J(\bar{\upsilon}) + \frac{1}{2}L_{\upsilon\upsilon}(\bar{\upsilon},\bar{\lambda})[\upsilon-\bar{\upsilon}]^2 + \varepsilon\|u-\bar{u}\|_{L^1(Q_\varepsilon)}.$$

We set $\delta u = u - \bar{u}$. Let us define δ_y to be weak solution of the linearied system

$$\delta y_t + \nu A\delta y + \nu C\delta y + B'(\bar{y})\delta y = \delta u, \delta y = 0.$$

When we use δ_y instead of $y-\bar{y}$, we make the small error

$$r_1 := (y-\bar{y}) - \delta y = S(u) - S(\bar{u}) - S'(\bar{u})(u-\bar{u}).$$

We know that the control-to-state mapping is Frechet differentiable, by Remark 3.1, the remainder term satisfies

$$\frac{\|r_1\|_W}{\|\delta u\|_q} \to 0 \text{ as } \|\delta u\|_q \to 0.$$

Substituting $y-\bar{y} = \delta y + r_1$, we obtain

$$L_{yy}(\bar{\upsilon},\bar{\lambda})[y-\bar{y}]^2 = L_{yy}(\bar{\upsilon},\bar{\lambda})[\delta y]^2 + 2L_{yy}(\bar{\upsilon},\bar{\lambda})[\delta y, r_1]$$
$$+L_{yy}(\bar{\upsilon},\bar{\lambda})[r_1]^2 = L_{yy}(\bar{\upsilon},\bar{\lambda})[\delta y]^2 + r_2.$$

The remainder term can be estimated by

$$|r_2| \le c(\|\delta y\|_W + \|r_1\|_W)\|r_1\|_W \le c(\|\delta u\|_q + \|r_1\|_W)\|r_1\|_W$$

and it follow that r_2 satisfies

$$\frac{|r_2|}{\|\delta u\|_q} \to 0 \text{ as } \|\delta u\|_q \to 0.$$

Let us abbreviate $\delta_v = (\delta_y, \delta_u)$. So far, we achieved the following estimate for the difference of the objective values.

$$J(\upsilon) - J(\bar{\upsilon}) \ge \frac{1}{2}L_{\upsilon\upsilon}(\bar{\upsilon},\bar{\lambda})[\delta\upsilon]^2 + \varepsilon\|u-\bar{u}\|_{L^1(Q_\varepsilon)} + r_2.$$

In the next step, we want to apply the coercivity assumption (SSC). To do this, we split δ_u in two components as follows:

$\delta u = h_u + r_u$

Where h_u and r_u are defined by

$$h_{u,i} = \begin{cases} 0 & on\ Q_{\varepsilon,i} \\ \delta u_i & on\ Q \mid Q_{\varepsilon,i}. \end{cases}$$

$$r_{u,i} = \begin{cases} \delta u_i & on\ Q_{\varepsilon,i} \\ 0 & on\ Q \mid Q_{\varepsilon,i}. \end{cases}$$

for all $i = 1,2$. Observe that h_u, r_u are orthogonal, i.e. $(h_u, r_u)_Q = 0$. Moreover, it follows from the definition that the identify

$$\|r_u\|_p = \|\delta u\|_{L^p(Q_\varepsilon)} = \|u - \bar{u}\|_{L^p(Q_\varepsilon)}.$$

Analogously, we split $\delta_y = h_y + r_y$, where h_y and r_y are solutions of the respective linearied system with right-hand sides h_u and r_u. Further, we set $h_v := (h_y, h_u)$ and $r_v := (r_y, r_u)$. We continue the investigation of the Lagrangian.

$$L_{yy}(\bar{\upsilon},\bar{\lambda})[\delta\upsilon]^2 = L_{\upsilon\upsilon}(\bar{\upsilon},\bar{\lambda})[h_\upsilon]^2 + 2L_{\upsilon\upsilon}(\bar{\upsilon},\bar{\lambda})[h_\upsilon, r_\upsilon] + L_{\upsilon\upsilon}(\bar{\upsilon},\bar{\lambda})[r_\upsilon]^2. \tag{17}$$

Now, we can use (SSC) to obtain

$$L_{\upsilon\upsilon}(\bar{\upsilon},\bar{\lambda})[h_\upsilon]^2 \ge \delta \|h_\upsilon\|_q^2. \tag{18}$$

The derivative L_{uu} can be splitted according to Theorem 3.2 into two addends, L_{uu} and L_{yy}, and no mixed derivatives appear. At first, we find

$$2L_{uu}(\bar{\upsilon},\bar{\lambda})[h_u, r_u] + L_{uu}(\bar{\upsilon},\bar{\lambda})[r_u]^2 = 2\gamma(h_u, r_u)_Q + \gamma\|r_u\|_2^2 \ge 0. \tag{19}$$

Secondly, we investigate

$$2L_{yy}(\bar{\upsilon},\bar{\lambda})[h_y, r_y] + L_{yy}(\bar{\upsilon},\bar{\lambda})[r_y]^2.$$

The following estimate is a conclusion of the inequality (16) and the Lipschitz continuity of the solution mapping of the linearied system

$$\begin{aligned} &\left|2L_{yy}(\bar{\upsilon},\bar{\lambda})[h_y, r_y] + L_{yy}(\bar{\upsilon},\bar{\lambda})[r_y]^2\right| \\ &\quad \ge -c\|r_y\|_W \left(\|h_y\|_W + \|r_y\|_W\right) \\ &\quad \ge -c\|r_u\|_q \left(\|h_u\|_q + \|r_u\|_q\right) \\ &\quad \ge -\frac{\delta}{2}\|h_u\|_q^2 - c\|r_u\|_q^2. \end{aligned} \tag{20}$$

Using the relation $\|h_u\|_q^2 \ge \frac{1}{2}\|\delta u\|_q^2 - \|r_u\|_q^2$,

we get by (17)-(20).

$$\begin{aligned} L_{\upsilon\upsilon}(\bar{\upsilon},\bar{\lambda})[\delta\upsilon]^2 &\ge \frac{\delta}{2}\|h_u\|_q^2 - c\|r_u\|_q^2 \\ &\ge \frac{\delta}{4}\|\delta u\|_q^2 - c\|r_u\|_q^2 \\ &= \frac{\delta}{4}\|u - \bar{u}\|_q^2 - c\|u - \bar{u}\|_{L^q(Q_\varepsilon)}^2. \end{aligned}$$

Moreover, we proved the following estimate

$$J(\upsilon) - J(\bar{\upsilon}) \ge \frac{\delta}{8}\|u - \bar{u}\|_q^2 + \left(\varepsilon - c\|u - \bar{u}\|_{L^s(Q_\varepsilon)}^2\right)\|u - \bar{u}\|_{L^1(Q_\varepsilon)}^2 + r_2.$$

By the interpolation estimate $\|u\|_q^2 \le \|u\|_1 \|u\|_s$, we get

$$J(\upsilon) - J(\bar{\upsilon}) \ge \frac{\delta}{8}\|u - \bar{u}\|_q^2 + \varepsilon\|u - \bar{u}\|_{L^1(Q_\varepsilon)}^2 - c\|u - \bar{u}\|_{L^1(Q_\varepsilon)}^2 + r_2.$$

We can choose ρ small enough, $\|u - \bar{u}\| \le \rho$, such that it holds $J(\upsilon) - J(\bar{\upsilon}) \ge \frac{\delta}{16}\|u - \bar{u}\|_q^2$.

Thus, we proved quadratic growth of the objective functions in a L^s- neighborhood of the reference control. It implies the local optimality of the pair $(\bar{y}, \bar{u})$. The proof is complete.

REFERENCES

[1]. C.T. Anh and D.T. Quyet (2012), *Long-time behavior for 2D non-autonomous g-Navier-Stokes equations*, Ann. Pol. Math. 103, 277-302.

[2]. C.T. Anh, D.T. Quyet and D.T. Tinh (2012), *Existence and nite time approximation of strong solutions of the 2D g-Navier-Stokes equations*, Acta. Math. Viet.

[3]. C.T. Anh and D.T. Quyet (2012), *g-Navier-Stokes equations with infinite delays*, Viet. J. Math., 40, 57-78.

[4]. D. Wachsmuth (2006), *Optimal control of the unsteady Navier-Stokes equations, Thesis, Berlin.*

[5]. H. Bae and J. Roh (2004), *Existence of solutions of the g-Navier-Stokes equations, Taiwanese J*, Math. 8, 85-102.

[6]. Nguyen Duc Loc (2016), *Necessary optimality conditions for the optimal control problem of g-Navier-Stokes equations*. Tạp chí Khoa học Kỹ thuật - Học viện Kỹ thuật Quân sự No2.

[7]. J. Roh (2005), *Dynamics of the g-Navier-Stokes equations*, J. Dffererential Equations 211, 452-484.

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