APPLICATION OF LAGRANGE MULTIPLIER METHOD IN THE PROBLEM OF FINDING THE MINIMUM AND MAXIMUM VALUES

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Abstract: *In this paper, we study the Lagrange multiplier method of the extrema of a multivariable function, from which we apply to solve problems of finding the maximum and minimum values of general math. Moreover, from the maximum value, the smallest value found, we can find a primary solution to the problem. In addition, we also create new math problems for secondary and high school students.*

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1. INTRODUCTION

In mathematics, the Lagrange multiplier method (named after the mathematician Joseph Louis Lagrange) is a method for finding the local minimum or maximum of a function of many variables satisfying certain given conditions. The problems of finding the maximum and the minimum value at the high school level also often have many variables, and often the variables satisfy some given condition. We find that the problem of finding the maximum value, the minimum value with many variables is like the problem of finding the conditional extrema of a function of many variables. Therefore, a question arises about whether it is possible to use the Lagrange multiplier method to solve problems about the maximum and minimum values?

On the other hand, extreme problems at the high school level are one of the most difficult types of math. It often appears in test questions for good students, entrance exams for grade 10, etc. Usually, we will find a solution. Of the problem if the extrema are known and the equal sign occurs. Therefore, we study and use the Lagrange multiplication method to find the extreme values, then we rely on the found extremes to find elementary solutions for these problems. To have a closer look at the Lagrange multiplication method, we refer readers to works [1, 2, 3] and the references therein.

2. CONTENT

2.1. Directional derivatives

Definition 2.1

The directional derivative of f at (x_0, y_0) in the direction of a unit vector $u = \langle a, b\rangle$ is

$$D_u f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h},$$

if this limit exists.

Theorem 2.2.[2,p. 912]

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $u = \langle a, b\rangle$ and

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b .$$

Definition 2.3.[2,p.913]

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by

$$\nabla f(x, y) = \left\langle f_x(x, y), f_y(x, y)\right\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} .$$

Definition 2.4.[2,p. 914]

The directional derivative of f at $(x_0; y_0; z_0)$ in the direction of a unit vector $u = \langle a, b, c\rangle$ is

$$D_u f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h},$$

if this limit exists.

If we use vector notation, then we can write both definitions (2.1 and 2.4) of the directional derivative in the compact form

$$D_u f(\mathbf{x}_0) = \lim_{h\to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h},$$

where $x_0 = \langle x_0, y_0\rangle$ if $n = 2$ and $x_0 = \langle x_0, y_0, z_0\rangle$ if $n = 3$. This is reasonable because the vector equation of the line through x_0 in the direction of the vector $\boldsymbol{u}$ is given by $x = x_0 + t\boldsymbol{u}$ and so $f(x_0 + h\boldsymbol{u})$ represents the value of f at a point on this line.

If $f(x, y, z)$ is differentiable and $\boldsymbol{u} = \langle a, b, c\rangle$, then the same method that was used to prove Theorem 2.2 can be used to show that

$$D_u f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c . \qquad (2.6)$$

For a function f of three variables, the gradient vector, denoted by ∇f or grad f, is

$$\nabla f(x,y,z) = \left\langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \right\rangle$$

or, for short,

$$\nabla f = \left\langle f_x, f_y, f_z \right\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Then, just as with functions of two variables, Formula 2.6 for the directional derivative can be rewritten as

$$D_{\mathbf{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}. \qquad (2.7)$$

2.2. Maximum and minimum values

One of the main uses of ordinary derivatives is in finding maximum and minimum values. In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables.

Look at the hills and valleys in the graph of f shown in Figure 1. There are two points (a,b) where f has a local maximum, that is, where $f(a,b)$ is larger than nearby values of $f(a,b)$. The larger of these two values is the absolute maximum. Likewise, f has two local minima, where $f(a,b)$ is smaller than nearby values. The smaller of these two values is the absolute minimum.

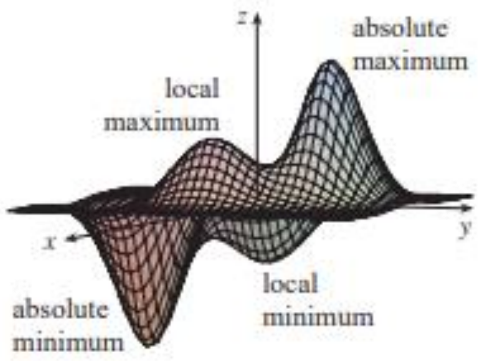

FIGURE 1

Definition 2.5.[2,p. 923]

A function of two variables has a local maximum at (a,b) if $f(x,y) \leq f(a,b)$ when is near (a,b). [This means that $f(x,y) \leq f(a,b)$ for all points (x,y) in some disk with center (a,b). The number $f(a,b)$ is called a local maximum value. If $f(x,y) \geq f(a,b)$ when (x,y) is near (a,b), then f has a local minimum at (a,b) and $f(a,b)$ is a local minimum value.

If the inequalities in Definition 2.5 hold for all points (x,y) in the domain of f, then f has an absolute maximum (or absolute minimum) at (a,b).

Theorem 2.6.[2,p.923]

If f has a local maximum or minimum at (a,b) and the first-order partial derivatives of f exist there, then $f_x(a,b)=0$ and $f_y(a,b)=0$.

2.2. Lagrange multipliers

It's easier to explain the geometric basis of Lagrange's method for functions of two variables. So we start by trying to find the extreme values of $f(x,y)$ subject to a constraint of the form $g(x,y)=k$. In other words, we seek the extreme values of $f(x,y)$ when the point (x,y) is restricted to lie on the level curve $g(x,y)=k$. Figure 1 shows this curve together with several level curves of f. These have the equations $f(x,y)=c$, where $c=7$, 8,9,10,11. To maximize $f(x,y)$ subject to $g(x,y)=k$ is to find the largest value of c such that the level curve $f(x,y)$ intersects $g(x,y)=k$. It appears from Figure 1 that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.) This means that the normal lines at the point (x_0,y_0) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$ for some scalar λ.

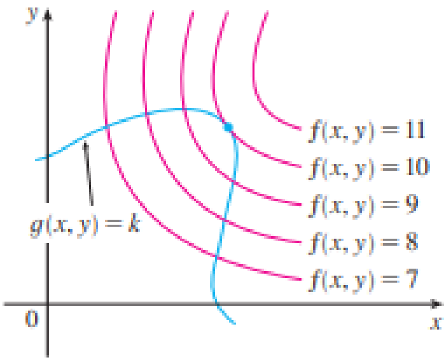

FIGURE 1

This kind of argument also applies to the problem of finding the extreme values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$. Thus the point (x,y,z) is restricted to lie on the level surface S with equation $g(x,y,z)=k$. Instead of the level curves in Figure 1, we consider the level surfaces $f(x,y,z)=c$ and argue that if the maximum value of f is $f(x_0,y_0,z_0)=c$, then the level surface $f(x,y,z)=c$ is tangent to the level surface $g(x,y,z)=k$ and so the corresponding gradient vectors are parallel.

This intuitive argument can be made precise as follows. Suppose that a function f has an extreme value at a point $P(x_0,y_0,z_0)$ on the surface S and let C be a curve with vector equation $r(t)=\langle x(t),y(t),z(t)\rangle$ that lies on S and passes through P. If t_0 is the parameter value corresponding to the point P, then $r(t_0)=\langle x_0,y_0,z_0\rangle$. The composite function $h(t)=f(x(t),y(t),z(t))$ represents the values that f takes on the curve C. Since f has an extreme value at (x_0,y_0,z_0), it follows that h has an extreme value at t_0, so $h'(t_0)=0$. But if f is differentiable, we can use the Chain Rule to write

$$0 = h'(t_0) = f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0)$$
$$= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0).$$

This shows that the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $r'(t_0)$ to every such curve *C*. But we already know that the gradient vector of $g, \nabla f(x_0, y_0, z_0)$, is also orthogonal to $r'(t_0)$ for every such curve. This means that the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. Therefore, if$\nabla g(x_0, y_0, z_0) \neq 0$, there is a number λ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0).$$

The number λ in Equation 1 is called a Lagrange multiplier. The procedure based on Equation 1 is as follows.

Method of Lagrange multiplers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$):

4. Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z).$$

And

$$g(x, y, z) = k.$$

(b) Evaluate $f$at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Example 2.5. [1,p. 149] Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint $x^2 + y^2 = 10$.

For this problem, $f(x, y) = 3x + y$ and $g(x, y) = x^2 + y^2 - 10$.

Let's go through the steps:

- $\nabla f = \langle 3,1 \rangle$.
- $\nabla g = \langle 2x, 2y \rangle$.

This gives us the following equation:

$$\langle 3,1 \rangle = \lambda \langle 2x, 2y \rangle.$$

We break up the above equation and consider the following system of 3 equations with 3 unknowns (x, y, λ).

Now, we plug λ back into our original equations and get $x = \mp 3$ and $y = \mp 1$. We get the following extreme points $(3,1)$; $(-3,-1)$.

We can classify them by simply finding their values when plugging into $f(x, y)$.

- $f(3,1) = 9 + 1 = 10$.
- $f(-3,-1) = -9 - 1 = -10$.

So the maximum happens at $(3,1)$ and the minimum happens at $(-3,-1)$.

2.3. Application of Lagrange Multipliers method in the problem of finding the Minimum and Maximum value

Problem 3.1 . Given positive real numbers x, y such that $x + y = 3$. Find the minimum value of the expression.

$$T = x^2 + 2y + \frac{25}{4y} - \frac{1}{2}.$$

Solution:

Application of Lagrange Multipliers:

We have: $f(x, y) = x^2 + 2y + \frac{25}{4y} - \frac{1}{2}$ and $g(x, y) = x + y - 3$.

Let's go through the steps:

- $\nabla f = \left\langle 2x, 2 - \frac{25}{4y^2} \right\rangle$.
- $\nabla g = \langle 1,1 \rangle$.

This gives us the following equation:

$$\Leftrightarrow \left\langle 2x, 2 - \frac{25}{4y^2} \right\rangle = \lambda \langle 1,1 \rangle.$$

So, we have the critical point $\left(\frac{1}{2}, \frac{5}{2}\right)$. Hence: $f\left(\frac{1}{2}, \frac{5}{2}\right) = \frac{29}{4}$.

Solving by the elementary method:

We have:

$$T = \left(x^2 - x + \frac{1}{4} \right) + y + \frac{25}{4y} + (x + y) - \frac{3}{4}.$$

$$T = \left(x - \frac{1}{2} \right)^2 + y + \frac{25}{4y} + (x + y) - \frac{3}{4}.$$

Moreover: $\left(x - \frac{1}{2} \right)^2 \geq 0$ for all x, so:

$$T \geq y + \frac{25}{4y} + (x+y) - \frac{3}{4}.$$

Application Cauchy's inequality to two positive numbers y and $\frac{25}{4y}$:

$$T \geq 2\sqrt{y.\frac{25}{4y}} + 3 - \frac{3}{4} = \frac{29}{4}.$$

Hence the minimum value of the expression is $\frac{29}{4}$ when $x = \frac{1}{2}, y = \frac{5}{2}$.

Problem 3.2. [TS 10 Vĩnh Long 2019-2020]

Given positive real numbers x, y such that $x + y = 1$. Find the minimum value of the expression.

$$A = 2x^2 - y^2 + x + \frac{1}{x} + 1.$$

Solution:

Application of Lagrange Multipliers

We have: $f(x, y) = 2x^2 - y^2 + x + \frac{1}{x} + 1$ and $g(x, y) = x + y - 1$.

Let's go through the steps:

- $\nabla f = \left\langle 4x + 1 - \frac{1}{x^2}, -2y \right\rangle$.
- $\nabla g = \langle 1,1 \rangle$.

This gives us the following equation:

$$\Leftrightarrow \left\langle 4x + 1 - \frac{1}{x^2}, -2y \right\rangle = \lambda \langle 1,1 \rangle.$$

$$\Leftrightarrow \begin{cases} 4x + 1 - \frac{1}{x^2} = \lambda \\ -2y = \lambda \\ x + y = 1 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2} \\ y = 1 \end{cases}.$$

So, we have the critical point $M(\frac{1}{2}, 1)$ of A.

Hence the minimum value of the expression is $\frac{15}{4}$ reach at $x = y = \frac{1}{2}$.

Solving by the elementary method:

Since $x + y = 1$ implies $y = 1 - x$, plugging into A, we have:

$$A = 2x^2 - (1-x)^2 + x + \frac{1}{x} + 1 = x^2 + 3x + \frac{1}{x}.$$

$$A=\left(x^2-x+\frac{1}{2}\right)+\left(4x+\frac{1}{x}\right)-\frac{1}{4}.$$

$$A=\left(x-\frac{1}{2}\right)^2+\left(4x+\frac{1}{x}\right)-\frac{1}{4}.$$

$$A\geq 0+2\sqrt{4x\cdot\frac{1}{x}}-\frac{1}{4}=\frac{15}{4}.$$

Hence the minimum value of the expression is $\frac{15}{4}$ reach at $x=y=\frac{1}{2}$.

Problem 3.3. Let $a,b>0$ such that $a+b\leq\frac{7}{2}ab$. Find the minimum value of the expression:

$$Q=9a+16b+\frac{1}{a}+\frac{1}{b}.$$

Solution:

We have: $a+b\leq\frac{7}{2}ab \Rightarrow \frac{1}{a}+\frac{1}{b}\leq\frac{7}{2}$.

We set: $x=\frac{1}{a}$, $y=\frac{1}{b}$ and $Q=x+y+\frac{9}{x}+\frac{16}{y}$.

Application of Lagrange Multipliers

We have: $f(x,y)=x+y+\frac{9}{x}+\frac{16}{y}$ and $g(x,y)=x+y-\frac{7}{2}$. .

Let's go through the steps :

- $\nabla f=\left\langle 1-\frac{9}{x^2},1-\frac{16}{y^2}\right\rangle$.
- $\nabla g=\langle 1,1\rangle$.

This gives us the following equation:

$$\left\langle 1-\frac{9}{x^2},1-\frac{16}{y^2}\right\rangle=\lambda\langle 1,1\rangle.$$

We consider the following system of 3 equations with 3 unknowns $x,y$and λ .

$$\begin{cases}1-\frac{9}{x^2}=\lambda\\ 1-\frac{16}{y^2}=\lambda\\ x+y=\frac{7}{2}\end{cases}\Rightarrow\begin{cases}\frac{3}{x}=\frac{4}{y}\\ x+y=\frac{7}{2}\end{cases}\Rightarrow\begin{cases}x=\frac{3}{2}\\ y=2\end{cases}.$$

So: $\left(\frac{3}{2},2\right)$ is the critical point and $f\left(\frac{3}{2},2\right)=\frac{35}{2}$.

Hence the minimum value of the expression is $\frac{35}{2}$ reached at $a = \frac{2}{3}$, $b = \frac{1}{2}$.

Solving by the elementary method:

We have: $Q = \left(4x + \frac{9}{x}\right) + \left(4y + \frac{16}{y}\right) - 3(x + y)$

Apply Cauchy's inequality to positive numbers $\left(4x, \frac{9}{x}\right)$ and $\left(4y, \frac{16}{y}\right)$:

$$Q \geq 2\sqrt{4x.\frac{9}{x}} + 2\sqrt{4y.\frac{16}{y}} - 3.\frac{7}{2} = \frac{35}{2} \ .$$

Hence the minimum value of the expression is $\frac{35}{2}$ reached at $a = \frac{2}{3}$, $b = \frac{1}{2}$.

Problem 3.4 Let $a, b, c > 0$ such that $\frac{7}{a} + \frac{28}{b} + \frac{108}{c} \leq 75$. Find the minimum value of the expression:

$$T = \frac{a}{2a+1} + \frac{b}{b+1} + \frac{7c}{2c+3}.$$

Solution:

We set: $x = \frac{1}{a}$, $y = \frac{1}{b}$, $z = \frac{1}{c}$ with $x, y, z > 0$.

$\Rightarrow 7x + 28y + 108z \leq 75$ and $T = \frac{1}{2+x} + \frac{1}{1+y} + \frac{7}{2+3z}$.

Application of Lagrange Multipliers

We get: $f(x, y, z) = \frac{1}{2+x} + \frac{1}{1+y} + \frac{7}{2+3z}$ and $g(x, y, z) = 7x + 28y + 108z - 75$.

Let's go through the steps:

- $\nabla f = \left\langle \frac{-1}{(2+x)^2}, \frac{-1}{(1+y)^2}, \frac{-21}{(2+3z)^2} \right\rangle$.
- $\nabla g = \langle 7, 28, 108 \rangle$.

This gives us the following equation:

$$\left\langle \frac{-1}{(2+x)^2}, \frac{-1}{(1+y)^2}, \frac{-21}{(2+3z)^2} \right\rangle = \lambda \langle 7, 28, 108 \rangle \ .$$

We consider the following system of 4 equations with 4 unknowns (x, y, z, λ):

$$\begin{cases} \frac{-1}{(2+x)^2} = 7\lambda \\ \frac{-1}{(1+y)^2} = 28\lambda \\ \frac{-21}{(2+3z)^2} = 108\lambda \\ 7x + 28y + 108z = 75 \end{cases} \Leftrightarrow \begin{cases} x + 2 = 2 + 2y \\ 7 + 7y = 6 + 9z \\ 7x + 28y + 108z = 75 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = \frac{1}{2} \\ z = \frac{1}{2} \end{cases} .$$

So we get $(x, y, z) = \left(1, \frac{1}{2}, \frac{1}{2}\right)$ is the critical point and $f\left(1, \frac{1}{2}, \frac{1}{2}\right) = 3$

Hence the minimum value of the expression is 3 reached at $a = 1, b = c = 2$.

Solution 2:

We have:

$$T = \frac{1}{9}(2+x) + \frac{1}{2+x} + \frac{4}{9}(1+y) + \frac{1}{1+y} + \frac{4}{7}(2+3z) + \frac{7}{2+3z} - \frac{1}{63}(7x+28y+108z) - \frac{38}{21}$$

Apply Cauchy's inequality, we get:

$$T \geq 2\sqrt{\frac{2+x}{9}.\frac{1}{2+x}} + 2\sqrt{\frac{4(1+y)}{9}.\frac{1}{1+y}} + 2\sqrt{\frac{4(2+3z)}{7}.\frac{7}{2+3z}} - \frac{1}{63}.75 - \frac{38}{21}.$$

$$T \geq \frac{2}{3} + \frac{4}{3} + 4 - \frac{75}{63} - \frac{38}{21} = 3.$$

Hence the minimum value of the expression is 3 reached at $a = 1, b = c = 2$.

3. CONCLUSION

The article mentions the application of The Lagrange multiplier method in solving conditional extrema problems. I give some problems close to high school students and innovative solutions. Moreover, it helps students practice thinking and creativity effectively. Besides, through the Lagrange method, we can find the elementary solution.

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ỨNG DỤNG PHƯƠNG PHÁP NHÂN TỬ LAGRANGE TRONG BÀI TOÁN TÌM GIÁ TRỊ LỚN NHẤT VÀ GIÁ TRỊ NHỎ NHẤT

Tóm tắt: *Trong bài báo này, chúng tôi nghiên cứu phương pháp nhân tử Lagrange của cực trị hàm nhiều biến, từ đó chúng tôi áp dụng giải các bài toán tìm giá trị lớn nhất, giá trị nhỏ nhất của toán phổ thông. Hơn nữa, từ giá trị lớn nhất, giá trị nhỏ nhất tìm được chúng tôi đưa ra lời giải sơ cấp cho bài toán. Ngoài ra chúng tôi cũng sáng tạo thêm các bài toán mới cho học sinh phổ thông.*

Từ khoá: *Bài toán GTLN, GTNN, phương pháp nhân tử Lagrange.*