SOME APPLICATION MODELS OF FIRST-ORDER DIFFERENTIAL EQUATIONS IN FACT

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Abstract: *Differential equations is a field of Mathematics that has many practical applications. This paper presents the application of first-order differential equations with five models: Model describing radioactive atoms, model for Spread of Disease, model of population growth, Model of bank interest rate, Criminal investigation with specific examples for readers to easily access.*
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1. INTRODUCTION

Differential equations is a field of mathematics that has always attracted the strong interest of mathematicians and applied scientists. The theory of differential equations becomes an effective tool, especially in describing and analyzing practical problems not only in science and technology but also in many different fields such as medicine, biology, economics, environment... The article presents examples to clarify some models of real-world problems for differential equations, helps readers better understand the role and application of differential equations in life.

2. CONTENT

2.1. Some basic concepts [1] [2]

2.1.1. Definitions

A first - order differential equation is an equation of the general form:

$$F\left(x, y, y'\right) = 0 \quad (1)$$

where the function F is defined in the domain $D \subset R^3$

If in domain D, from equation (1) we can solve y':

$$y' = f\left(x, y\right)$$

then we get the first - order differential equation solved to the derivative.

2.1.2. Solution of the differential equation

The function $y=\varphi(x)$ is determined and differentiable on the interval $I=(a;b)$ is called a solution of equation (1) if the following two conditions are satisfied:

a) $(x,\varphi(x),\varphi'(x))\in D \;\; \forall \;\; x\in I$

b) $F(x,\varphi(x),\varphi'(x))\equiv 0$ in the domain I

2.1.3. The Existence and Uniqueness Theorem

Cauchy Problem: The solution of a first order differential equation is infinite. The set of solutions of a first order differential equation depends on an arbitrary constant *c*. In practice, people are often interested in solutions of first - order differential equations that satisfy certain condition**s.**

We consider the following problem for equation (2), which is called the Cauchy problem (or initial value problem):

Find $y(x)$ *satisfied:* $\begin{cases} y'=f(x,y) \\ y(x_0)=y_0 \end{cases}$ *(2) where* $(x_0,y_0)\in D$ *is called the initial condition.*

Lipschitz's condition: Let the function $f(x,y)$ is defined on a set $D\subset R^2$. Function *f* satisfies a **Lipschitz condition** in the variable y on a set *D* if a constant $L>0$ exists with:

$$|f(x,y_1)-f(x,y_2)|\le L|y_1-y_2|$$

whenever $(x,y_1),(x,y_2)\in D$. The constant L is called a Lipschitz constant for f .

The Existence and Uniqueness Theorem: Suppose the function $f(x,y)$ in (2) is continuous and satisfies the Lipschitz condition for the variable y on the rectangle:

$$D=\{(x,y)\in R^2 \,/\, |x-x_0|\le a, |y-y_0|\le b\}$$

Then the solution of Cauchy's problem exists and is unique in the interval $I=[x_0-h,x_0+h]$ with $h=\min\left(a,\dfrac{b}{M}\right)$ and $M=\max\limits_{(x,y)\in D}|f(x,y)|$.

2.1.4. Separable differential equations

Separable differential equations is a differential equation that has one of the following forms:

$$\frac{dy}{dx}=\frac{f(x)}{g(y)},\;\; \frac{dy}{dx}=\frac{g(y)}{f(x)} \text{ or } \frac{dy}{dx}=f(x)g(y)$$

where $f(x),g(y)$ are continuous functions on $(a,b)\subset R$.

Solution: We put the differential equation in the form: $f(x)dx+g(y)dy=0$

Integrate both sides, then the general solution of the equation is:

$$\int f(x)dx+\int g(y)dy=C$$

2.2. Application models

2.2.1. Model describing radioactive atoms

Let R_0 be the mass of radium initially at $t=0$ and R(t) be its mass at any time t.

Then the decay rate is $\frac{dR}{dt}$. This speed is a negative quantity because R decreases over time

According to the problem condition, we have:

$\frac{dR}{dt}=-kR,\left(R>0\right)$ where k is the scaling factor $\left(k>0\right)$

Integrating the above equation, we have:

$\ln R=-kt+\ln\left|C_1\right|$ hay $R\left(t\right)=C.e^{-kt}\left(C=\left|C_1\right|\right)$

I see that $R=R_0$ when the $t=0$

Substituting these values into the expression of the general solution we find $C=R_0$

Therefore, $R\left(t\right)=R_0e^{-kt}$

Example 1: It is known that the decay rate of radium is directly proportional to its quantity at each given instant. Find the decay law of radium, given its initial mass and the time T needed to decay half of the original radium mass. The half – life of radium is 1600 years. What percentage of a given amount of the radium will remain after 100 years?

According to the problem condition $R=\frac{R_0}{2}$ when $t=T$. We have k:

$$k=\frac{\ln\frac{1}{2}}{-T}=\frac{\ln 2}{T}$$

So, the decay law of radium is expressed by the formula $R=R_0e^{-\frac{\ln 2}{T}t}$

Therefore: $R(100)=R_0e^{-0{,}043t}$

$$R(100)=e^{-0{,}043}=0.958=95.8\%$$

So after 100 years, 4.2 percent of radium will be remain.

2.2.2. Model for Spread of Disease

The rate of spread is proportional to the number of people infected and the number of people who are not infected.

Let $N\left(t\right)$ be the number of infected people at time t, P be the total number of people (constant); k is the growth constant. Because the rate of spread is proportional to the number of infected and uninfected people, we have the equation

$$\frac{dN(t)}{dt}=kPN(t)-kN^2(t)$$

Solving the differential equation, we have

$$\frac{dN(t)}{dt} = kPN(t) - kN^2(t) \Leftrightarrow \frac{dN(t)}{N(t).(P - N(t))} = kdt$$

Integrating the above equation, we have:

$$\frac{1}{P}\int\left(\frac{1}{N(t)} + \frac{1}{P - N(t)}\right)dN(t) = \int kdt \Leftrightarrow \frac{1}{P}\ln\left|\frac{N(t)}{P - N(t)}\right| = kT + C$$

$$\Leftrightarrow \ln\left|\frac{N(t)}{P - N(t)}\right| = Pkt + C \Leftrightarrow \frac{N(t)}{P - N(t)} = e^{Pkt}.C$$

So $N(t) = \dfrac{P}{1 + C.e^{-Pkt}}$

Example 2: There is a model of the spread of the Covid - 19, in which the rate of spread is proportional to the number of infected and uninfected people. In a remote town of 5000 residents, the number of people with covid at the beginning of the week was 160 and this number had increased to 1200 by the end of the week. How long does it take for 80% of the town's residents to be infected with Covid -19?.

With $P = 5000$ and $N(0) = 160$, we have $160 = \dfrac{5000}{1 + C.e^0} \Rightarrow C = \dfrac{121}{4}$

Because after 7 days the infected population increases to 1200, we have:

$$1200 = \frac{5000}{1 + C.e^{-k.5000.7}} \Rightarrow k = \frac{1}{35000}\ln\left(\frac{363}{38}\right)$$

Because 80% of the inhabitants of the town are infected, the equation is as follows

$$80\%.5000 = \frac{5000}{1 + C.e^{-k.5000.t}} \Rightarrow t = 14.875$$

Thus, after about 15 days, 80% of the town's residents were infected with Covid -19.

2.2.3. Model of population growth [3]

The logistic equation was introduced (around 1840) by the Belgian mathematician and demographer **P.F. Verhulst** as a possible model for human population growth.

Where *P(t)* is the number of people at time *t* , *K* is the capacity of the medium, *r* is a constant.

(**Note**: K is called the carrying capacity if it is the maximum population that the environment can sustain over a long period of time)

Using the description by the logistic function, we have a mathematical model that predicts population growth as follows:

$$\frac{dP}{dt} = rP(t)\left(1-\frac{P(t)}{K}\right) \text{ or } P'(t) = rP(t)\left(1-\frac{P(t)}{K}\right)$$

Solving this differential equation by the method of dissociation of variables with P_0 being the number of people at time *t = 0* (or the number of people at the beginning of time *t*), we get:

$$\frac{dP}{dt} = rP(t)\left(1-\frac{P(t)}{K}\right) \Leftrightarrow \int \frac{dP}{rP(t)\left(1-\frac{P(t)}{K}\right)} = \int dt \Leftrightarrow \frac{K}{r}\int \frac{dP}{P(t)\left(K-P(t)\right)} = \int dt$$

$$\Leftrightarrow \ln\left|\frac{P(t)}{P(t)-K}\right| = rt + C \text{ or } P(t) = \frac{K}{1+C.e^{-rt}}$$

Example 3: In 1990, the world population was about 5.28 billion people and in 2000, the world population reached 6.07 billion people. Assume that the relative growth rate decreases as the population increases and starts to become negative when the population size exceeds its capacity to accommodate K, which is the maximum population the environment can tolerate in the long run (also called the logistic model). Assuming the capacity is 100 billion people, use a logistic model to predict the world population in 2025?

Because the capacity is 100 billion people, K = 100 billion

In 1990, the world population was 5.28 billion, so for $t = 0$ then $P_0 = 5,28$ tỉ

Applying the results of the general problem $P(t) = \frac{K}{1+C.e^{-rt}}$ with $t = 0, P(t) = P_0 = 5,28$ billion and K = 100 billion, we have $5,28 = \frac{100}{1+C.e^{0}} \Rightarrow C \approx 18$

In 2000, the world population was 6,07 billion, so for $t = 10$ then $P_1 = 6,27$ tỉ

With $t = 10, P(t) = P_1 = 6,27$ billion, $C = 18$ and K = 100 billion, we have:

$$P(t) = \frac{K}{1+C.e^{-rt}} \Leftrightarrow 6,07 = \frac{100}{1+18.e^{-10r}} \Leftrightarrow r \approx 0,015$$

In the year 2025, we have t = 35, , $C = 18$ and K = 100 billion, we have:

$$P(t) = \frac{K}{1+C.e^{-rt}} \Leftrightarrow P\left(35\right) = \frac{100}{1+18.e^{-0,015.35}} \approx 8,551 \text{ billion people}$$

So, in 2025, the world population is estimated at 8.551 billion people.

2.2.4. Model of bank interest rate

Problem 1: Suppose you initially have an amount S_0 of millions. You make a plan to deposit that money in the bank at a r% interest rate for one year. What is the amount received after t years?

The symbol S(t) is the amount of money you have after a period of t years. We have: $S(t+\Delta t)=S(t)+r\%\Delta tS(t)$ where $r\Delta tS(t)$ is the amount of interest generated after a period of Δt, $k\Delta t$ is the amount you pay in addition.

Then $\frac{S(t+\Delta t)-S(t)}{\Delta t}=r\%.S(t)$

Let $\Delta t\to 0$ we get $S'(t)=r\%.S(t)$

$$\Leftrightarrow \frac{dS(t)}{dt}=r\%.S(t)\Leftrightarrow \frac{dS(t)}{S(t)}=r\%dt$$

Integrating both sides we get $\ln\left|S(t)\right|=r\%t+C\Leftrightarrow S(t)=e^{r\%t+C}=C.e^{r\%t}$

Solving the differential equation with the initial condition $S(0)=S_0$, we have $S(t)=S_0e^{r\%t}$

Example 4.1: Nam deposits an amount of VND 8 million in a bank with an interest rate of 0.9% for 1 month. After exactly 5 years, how much is the amount in the book knowing that during that time Nam did not withdraw a single penny.

With an interest rate of 0.9% for 1 month, we will have 10.8% for one year.

According to the formulation of problem 1, we have

$$S(5)=8.10^6.e^{10,8\%.5}=13728054,9 \text{ million}$$

Problem 2: Suppose you plan to save money for the future.

Every year you deposit k million and you deposit it in the bank with compound interest of $r\%$ for one year (unchanged). We will construct the initial value versus balance over time problem as follows:

The symbol S(t) is the amount of money you have after a period of t years. Then $S(t+\Delta t)=S(t)+r\%\Delta tS(t)+k\Delta t$ where $r\%\Delta tS(t)$ is the amount of interest generated after the time Δt, $k\Delta t$ is the amount you pay in addition.

Then $\frac{S(t+\Delta t)-S(t)}{\Delta t}=r\%S(t)+k$

Let $\Delta t\to 0$ we get $S'(t)=r\%.S(t)+k$

$$\Leftrightarrow \frac{dS(t)}{dt}=r\%.S(t)+k\Leftrightarrow \frac{dS(t)}{r\%.S(t)+k}=dt$$

Integrating both sides, we get:

$$\frac{1}{r\%}\ln\left|r\%S\left(t\right)+k\right|=t+C\Leftrightarrow r\%S\left(t\right)+k=e^{r\%t+C}\Leftrightarrow S\left(t\right)=\frac{1}{r\%}\left(Ce^{r\%t}-k\right)$$

Solving the differential equation with the initial condition $S\left(0\right)=0$, we have

$$S\left(t\right)=\frac{k}{r\%}e^{r\%t}-\frac{k}{r\%}$$

Example 4.2: Suppose at the age of 30, you plan to save money for your children, every year you deposit *k* million and you deposit it in a bank with an interest rate of *6%* for one year (unchanged), you retire at the age of 65. For you to have at least a billion in savings for your children when you retire, how much is k equal to?

According to the formula in problem 2, we have:

$$S\left(t\right)\geq 10^9\Leftrightarrow\frac{k}{6\%}e^{6\%.35}-\frac{k}{6\%}\geq 10^9\Leftrightarrow k\geq 8372673.371$$

So, every year, you need to deposit at least 8372673.371

2.2.5. Criminal investigation [3]

Newton's law of cooling: The rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its environment.

It is assumed that the body temperature $T\left(t\right)$ obeys the law of cooling with the expression $\frac{dT\left(t\right)}{dt}=k\left(T\left(t\right)-M\right)$ where T is the body temperature at time *t* (hours), M is the constant temperature of the environment, and *k* is a constant that depends on the material properties of the object.

Solving the differential equation, we have:

$$\frac{dT\left(t\right)}{T\left(t\right)-M}=kdt\Leftrightarrow\ln\left|T\left(t\right)-M\right|=kt+C\Leftrightarrow T\left(t\right)=Ce^{kt}+M$$

Scope of application of the law: This law has many applications ranging from determining the time it takes for a cup of coffee to cool to a drinkable temperature, to its use in forensic science, to determine how long a corpse has been dead**.**

Example 5: At one house, a victim was killed. Police were called at 10:56 am. The police took the victim's body temperature at that time was 31^0. An hour later, the police took the victim's temperature again and the victim's body temperature at that time was 30^0. The temperature in the house was 20^0. What time was the victim killed?

We have:

$$\begin{cases}T\left(0\right)=31\\T\left(1\right)=30\end{cases}\Leftrightarrow\begin{cases}31=C+22\\30=C.e^k+22\end{cases}\Leftrightarrow\begin{cases}C=9\\k=-0,11778\end{cases}.$$

So $T(t)=9.e^{kt}+22$

When the victim is killed, the victim's body temperature is 37^0, We have the time the victim is killed by the time the police arrive: $t = \frac{\ln 15 - \ln 9}{k} = -4,337$ (hours) = 4 hours 20 minutes. So the victim was killed at 6:36

3. CONCLUSION

Thus, in this article, I have presented a number of applications of differential equations in the fields of economics, demography, medicine, etc. From there, it helps readers see the relationship of mathematics in general, differential equations in particular to other sciences. In each model, the article has shown the relationship between the factors in the problem to lead to differential equations, solving differential equations along with specific problems so that readers have a better approach.

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MỘT SỐ MÔ HÌNH ỨNG DỤNG CỦA PHƯƠNG TRÌNH VI PHÂN CẤP MỘT TRONG THỰC TẾ

Tóm tắt: *Phương trình vi phân là một lĩnh vực Toán học có rất nhiều ứng dụng trong thực tế.* *Trong bài báo này tôi trình bày ứng dụng của phương trình vi phân cấp một với năm mô hình: Mô hình mô tả lượng nguyên tử phóng xạ, mô hình lây lan dịch bệnh, mô hình tốc độ tăng trưởng dân số, mô hình lãi suất ngân hàng, mô hình sử dụng trong điều tra tội phạm với các ví dụ cụ thể để người đọc có thể tiếp cận một cách dễ dàng.*

Từ khoá*: Tốc độ phân rã, lây lan dịch bệnh, tăng trưởng dân số, lãi suất, điều tra.*