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Analysis of the bar's free vibrations with considering lateral shear strain by the finite element method

Phân tích dao động tự do của thanh có xét đến biến dạng trượt ngang bằng phương pháp phần tử hữu hạn

> **A. Prof. Phd DOAN VAN DUAN**
Faculty of Engineering - Vietnam Maritime University.
Email: duandv.ct@vimaru.edu.vn

ABSTRACT:
The beam structure with a large cross-sectional height compared to the beam span, h/l≥l/5 (high beam) has been studied and widely applied in the fields of industrial, civil construction, traffic and irrigation. ..especially for high-rise buildings using vertical combined structural solutions (the lower floors are for showrooms, shops... need large space, while the upper floors are for hotels and houses ... only need small and medium space), in which people often use the structure of transfer floor, transfer truss or transfer beam with large cross-sectional height to transmit the load from the structure above to the foundation, because their ability to span large spans, they increase space, reduce the number of columns and create architecture for the building, meeting the practical needs of people. The Euler - Bernoulli beam theory is commonly used today, ignoring the effect of lateral shear strain caused by shear force, this is only true for beams with a small cross-sectional height compared to the beam length (h/L<l/ 5), but for high beams, the Euler-Bernoulli beam theory is no longer true. Therefore, in this paper, the author uses the forced displacement method combined with the finite element method to study the free vibration of the bar with different boundary conditions with considering the influence of the shear strain, the theory used here is the full beam theory [5]. The research results show that the influence of the lateral shear strain on the natural frequency of the bar is very large, for example, for the bar with one end fixed and the other pinned , the effect of the lateral shear strain is reduced by 34,075%, 38.707% compared to when the effect of lateral shear strain is not taken into account.

Keywords: finite element; oscillate; vibration; oscillation...Phần tử hữu hạn, dao động...

1. INTRODUCTION

The problem of eigenvalues and eigenvectors has been studied by many domestic and foreign scientists, but the current commonly used method is to bring the coefficient matrix of the equation of stability and free vibration of the bar to the diagonal form or band matrix form, strip along the main diagonal by different algorithms, such as Jacobi algorithm [8], LR [8], [10], QR[10], subspace [10]. ...which is very complicated, to get the product of that term gives us the characteristic polynomial equation to determine the eigenvalues. Although the methods [8], [10] have to transform the complex matrix, sometimes the solution is not reliable enough because the convergence of the problem depends on the properties of the matrix, symmetry or not symmetry, positive definite or not positive... traditional methods, such as Rayleigh's method [2], only give us the fundamental frequency of oscillation. Unlike foreign authors, some domestic authors have used forced displacement [3], [4], [5], [6] to find solutions for some other eigenvalue problems. for example, in [3], [4], [5] the authors use forced displacement method for the problem of vibration and stability of the bar structure, in [6] the author uses the method of Forced displacement method for the vibration problem of cable structures, according to the semi-analytic solution.

The forced displacement method has a simple and easy-to-understand view, by clicking the displacement at any point on the bar, it allows us to bring the eigenvalues of the freely oscillating bar to the differential equation on the right side, Solving this equation we immediately get the bar vibration frequencies without going through complex matrix transformations. Therefore, in this paper, the author also uses the above-mentioned forced displacement method, combined with the finite element method to build and solve the problem of free oscillation of the bar with considering the influence of the lateral shear strain according to the numerical solution.

2. THE PROBLEM OF FREE VIBRATION OF THE BAR WITH CONSIDERING THE LATERAL SHEAR STRAIN

Consider a straight bar, of constant cross-section, with mass m uniformly distributed over the bar. When there is a lateral displacement, then in addition to the internal forces M and Q, the inertia force f_m must also be considered. The force of inertia f_m is the product of the mass and the acceleration of motion and whose direction of action is the direction of motion (the direction of deflection) of the bar. Thus, the inertial force has the same effect as the lateral force, in this case is the distributed lateral force, applied at the bar axis. If the mass m is distributed over the height of the bar section, then due to the rotation of the bar cross section, there is also a rotational inertia force of the bar cross section. For simplicity in studying, we do not consider this rotational inertia force.

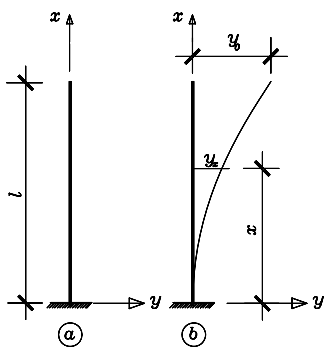


Figure 1. The bar with one end fixed and the othe free

With D'Alambert's principle, consider the force of inertia f_m as the external resistance force acting on the bar, and since the force of inertia is a function of time, the deflection and internal force functions in the bar are both functions of coordinates and time: W=W(x,t) is a function of deflection, M=M(x,t) is a function of bending moment, V=V(x,t) is a function of shear force.

The inertia force of the bar is calculated as follows:

$$f_m = m\frac{\partial^2 W}{\partial t^2} \qquad (1)$$

Considering the force of inertia f_m as a distributed external resistance force acting on the bar, immediately write two balanced differential equations

$$\left.\begin{aligned} &-\frac{\partial^2 M}{\partial x^2} + f_m = 0 \quad (a)\\ &-\frac{\partial M}{\partial x} + V = 0 \quad (b)\end{aligned}\right\} \qquad (2)$$

When considering the shear strain in the bar, the shear strain γ, the angle of rotation due to the bending moment θ, the bending strain χ and the internal moment force M are determined according to the following expressions:

$$\left.\begin{aligned} &\gamma = \frac{\alpha}{GF}V;\ \ \theta = \frac{\partial W}{\partial x} - \gamma\\ &\chi = -\frac{\partial^2 W}{\partial x^2} + \frac{\alpha}{GF}\frac{\partial V}{\partial x}\\ &M = -EJ\chi\end{aligned}\right\} \qquad (3)$$

Putting expressions (1) and (3) in (2) get

$$\left.\begin{aligned} &EJ\left(\frac{\partial^4 W}{\partial x^4} - \frac{\alpha}{GF}\frac{\partial^3 V}{\partial x^3}\right) + m\frac{\partial^2 W}{\partial t^2} = 0 \quad (a)\\ &EJ\left(\frac{\partial^3 W}{\partial x^3} - \frac{\alpha}{GF}\frac{\partial^2 V}{\partial x^2}\right) + V = 0 \quad (b)\end{aligned}\right\} \qquad (4)$$

The solution of system (4) can be written in the form

$$\left.\begin{aligned} &W(x,t) = y(x)\cos(\omega t) = y\cos(\omega t)\\ &V(x,t) = Q(x)\cos(\omega t) = Q\cos(\omega t)\end{aligned}\right\} \qquad (5)$$

Then system (4) has the form

$$\left.\begin{aligned} &\left(EJ(\frac{d^4 y}{dx^4} - \frac{\alpha}{GF}\frac{d^3 Q}{dx^3}) - m\omega^2 y\right)\cos(\omega t) = 0\\ &\left(EJ(\frac{d^3 y}{dx^3} - \frac{\alpha}{GF}\frac{d^2 Q}{dx^2}) + Q\right)\cos(\omega t) = 0\end{aligned}\right\} \qquad (6)$$

Since the component in brackets does not depend on t, system (6) is simplified as follows

$$\left.\begin{aligned} &EJ(\frac{d^4 y}{dx^4} - \frac{\alpha}{GF}\frac{d^3 Q}{dx^3}) - m\omega^2 y = 0\\ &EJ(\frac{d^3 y}{dx^3} - \frac{\alpha}{GF}\frac{d^2 Q}{dx^2}) + Q = 0\end{aligned}\right\} \qquad (7)$$

or is

$$\left.\begin{aligned} &EJ\frac{d^4 y}{dx^4} - \frac{\alpha h^2}{6}\frac{d^3 Q}{dx^3} - m\omega^2 y = 0\\ &EJ\frac{d^3 y}{dx^3} - \frac{\alpha h^2}{6}\frac{d^2 Q}{dx^2} + Q = 0\end{aligned}\right\} \qquad (7a)$$

The two functions y=y(x) and Q=Q(x) are both functions of the x coordinate. System (7) does not depend on the variable t, is a system of two linear differential equations with constant coefficients. When the shear strain is not considered, for $G\to\infty$ or for $h\to 0$, the first two equations of the system (7) and the system (7a) become the equations of vibration of the bar according to the Euler-Bernoulli beam theory, solving this equation to find deflection y and then use the second equation to calculate Q.

The general method for solving system (7) is to solve their two characteristic equations and construct the solutions y and Q on the basis of the solutions (eigenvalues) of the characteristic equations. However, we will use forced displacement method to solve.

3. THE FORCED DISPLACEMENT METHOD

When building the problem according to the method of Gaussian extremum principle, it is possible to use variable quantities (virtual displacement and virtual strain) that are independent of time.

$$\left.\begin{aligned} &\gamma_x = \frac{\alpha}{GF}Q;\ \theta_x = \frac{\partial y}{\partial x} - \gamma_x\\ &\chi_x = -\frac{\partial^2 y}{\partial x^2} + \frac{\alpha}{GF}\frac{\partial Q}{\partial x}\\ &M_x = EJ\chi_x\end{aligned}\right\} \qquad (8)$$

The letter x at the foot of quantities indicates that the quantity depends only on x.

The problem of free oscillation of the bar is referred to the problem of finding the minimum of the amount of coercion at any time t:

$$Z = \int_0^l M[\chi_x]dx + \int_0^l V[\gamma_x]dx + \int_0^l f_m[y]dx \to \min \qquad (9)$$

The quantity in square brackets of the functional (9) is a variable quantity.

From the minimum condition

$$\delta Z=\int_0^1 M\delta[\chi]dx+\int_0^1 V\delta[\gamma]dx+\int_0^1 f_m\delta[W]dx=0 \quad (10)$$

and using the differential calculus will get back two equations (6) and since the problem is linear in terms of t, it has system (7).

Thus, the problem of free oscillation of the bar using transform (5) leads to the solution of system (7) which does not contain the variable t. The y≠0 (non-trivial) solution of system (7) depends on the parameters m, EJ, ω and bar length. Usually, the parameters m, EJ and bar length are known so frequency is a function of these quantities.

Using quantities that do not contain a time variable t, problem (9) has the form

$$Z=\int_0^1 M_x[\chi_x]dx+\int_0^1 Q[\gamma_x]dx+\int_0^1 f_x[y]dx\to \min \quad (11)$$

here $M_x=EJ\chi_x,\ f_x=-m\omega^2 y$ (12)

To solve problem (11) we use forced displacement method by giving a certain point of the bar, for example point x1 , forced displacement y0.

$$g_1=y(x_1)-y_0=0 \quad (13)$$

The minimum problem (11) with constraint (13) is a static problem of calculating the bar subjected to forced displacement at the point x1, whose hidden is the frequency ω, so it can be called the free oscillation problem of the bar. Writing the extended Lagrange function F of (11) and (13), we have the extreme condition

$$\left.\begin{aligned}&\delta F=\int_0^1 M_x\delta\left[-\frac{d^2y}{dx^2}+\frac{\alpha}{GF}\frac{dQ}{dx}\right]dx+\int_0^1 Q\delta\left[\frac{\alpha}{GF}Q\right]dx\\&+\int_0^1 f_x\delta[y]dx+\delta[\lambda g_1]=0\end{aligned}\right\} \quad (14)$$

λ in (14) is the Lagrange factor and is the new unknown of the problem. From (14) get two balanced equations (two Euler equations):

$$\left.\begin{aligned}&EJ\left(\frac{d^4y}{dx^4}-\frac{\alpha}{GF}\frac{d^3Q}{dx^3}\right)-m\omega^2y=\begin{cases}-\lambda, x=x_1\\0, x\neq x_1\end{cases}\\&EJ\left(\frac{d^3y}{dx^3}-\frac{\alpha}{GF}\frac{d^2Q}{dx^2}\right)+Q=0\end{aligned}\right\} \quad (15)$$

along with equation (13). The system of equations (15) has the right side λ.

Mechanically, λ has the dimension is force and it is the holding force to displace at the point x=x1 of the bar by the forced displacement y0 (equation (13)). The holding force we put in so it has to equal zero. Mathematically, the equation of oscillation is an equation that has no right side (system (7)) so it must also be zero. So we have

$$\lambda=0 \quad (16)$$

The solution of equation (16) is also a solution of the left side (15) or of the system (7). Thus, equation (16) is a polynomial equation determining eigenvalues, when the functions y(x) and Q(x) satisfy the boundary conditions, it is a polynomial equation determining the eigenfrequency of the free vibration of the bar. In this case λ is the function of ω, λ=λ(ω).

The problem of free oscillation of the bar is reduced to problem (11) with constraint (13) and will be solved directly on the extended Lagrange functional to find the function λ(ω), solving equation (16) will get the frequencies eigenvalues, similar to the problem of determining the critical force of the bar [4]. Note, λ is the Lagrange factor of the constraint (13).

We are considering the case of uniformly distributed mass on the bar. The problem has infinitely many degrees of freedom, so there are infinitely many eigen frequencies. They form the oscillation natural frequency range of the bar whose lower boundary is the fundamental frequency and the upper boundary is infinitely large, ω→∞ . Bars with different boundary conditions will oscillate with different natural frequencies. The free oscillation natural frequencies of bars with different boundary conditions are calculated by the forced displacement method shown below.

4. THE PROBLEM OF FREE VIBRATION OF THE BAR - NUMERICAL SOLUTION

4.1. The finite element method

The finite element method divides the work into small parts called elements, the calculation of the work is led to the calculation of the small elements and then connects those elements together, we get the solution of a complete work. The interpolation function is chosen so that the calculation result is stable: the result is unique, a small change of the boundary condition or the initial condition does not change the calculation result.

The beam theory considering the influence of lateral shear deformation presented in [5] considers the deflection y shear force Q of the beam to be two functions to be determined, so it is necessary to define two interpolation functions for the above two hidden functions.

Based on the interpolation function, it is possible to calculate the stress and displacement fields of each element and thus establish the element stiffness matrix. Based on the element stiffness matrix, the overall stiffness matrix of the building is built.

Normally, for flexural beam elements, a third degree polynomial is used to describe the displacement.

$$y=a_0+a_1x+a_2x^2+a_3x^3 \quad (17)$$

We see that there are 4 parameters that need to be determined. However, for convenience, we replace 4 parameters a_0, a_1, a_2, a_3 with displacement, rotation angle of the two-nodes element as shown in Figure 2.

Due to the use of the 3rd order function, the forces acting on the element must all be reduced to the node, including the inertial force in the dynamic problem.

a. Bending element interpolation function

For flexural elements such as bars, a cubic polynomial is often used to calculate its displacement, so there are four parameters to be determined. It is possible to select a two-node element, each node has two parameters: displacement W and rotation angle θ at that node, figure 2.

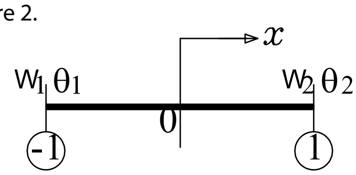


Figure 2. Two-node displacement element

For a general calculation, the element length is taken in two units, the origin is placed in the middle. Thus, if the parameters W_1,W_2, θ_1, θ_2 are known, the displacement of each point in the element is determined by the following cubic polynomial.

$$W(x)=f_1W_1+f_2W_2+f_3\theta_1+f_3\theta_2 \quad (18)$$

where

$$\left.\begin{aligned} f_1 &= \frac{1}{4}(x-1)^2(x+2);\ f_2 = \frac{1}{4}(x+1)^2(-x+2) \\ f_3 &= \frac{1}{4}(x-1)^2(x+1);\ f_4 = \frac{1}{4}(x+1)^2(x-1) \end{aligned}\right\}$$

We use the first degree polynomial to approximate the shear force function of the element, the shear force element contains two nodes, figure 3, each node has an unknown parameter Q_i is the element shear force at that position.

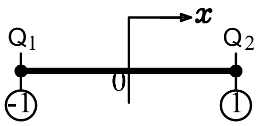


Figure 3. Two-nodes shear force element

The element length is taken by two units, the origin is placed in the middle of the element. If the shear forces Q_1, Q_2, at two nodes are known, then the shear force V at any point of the element is calculated by the formula.

$$V(x) = f_5Q_1 + f_6Q_2 \qquad (19)$$

where: $f_5 = \frac{1}{2}(1-x);\ f_3 = \frac{1}{2}(1+x)$

Thus, each element has two displacements of nodes W_1, W_2 two rotation angles θ_1, θ_2 and two shear forces of nodes Q_1, Q_2, a total of six parameters (6 hidden) to be determined.

Let's call {X} is the column vector containing the six hidden elements of the element in the following order.

$$\{X\} = [W_1\ \ W_2\ \ \theta_1\ \ \theta_2\ \ Q_1\ \ Q_2]^T \qquad (20)$$

then we can rewrite the expressions (10) and (11) in matrix form as follows.

$$\begin{aligned} W(x) &= [f_1\ \ f_2\ \ f_3\ \ f_4\ \ 0\ \ 0]\{X\} \\ V(x) &= [0\ \ 0\ \ 0\ \ 0\ \ f_5\ \ f_6]\{X\} \end{aligned} \qquad (21)$$

Sau khi đã biết các hàm chuyển vị và hàm lực cắt thì dễ dàng tính được biến dạng uốn χ_x, nội lực mômen M_x, biến dạng trượt γ_x, góc xoay φ (do mômen gây ra) của phần tử như sau. After knowing the displacement and shear force functions, it is easy to calculate the bending strain χ_x, internal moment force M_x, shear strain γ_x, and rotation angle φ (caused by moment) of the element as follows.

$$\chi_x = \left[-\frac{d^2W}{dx^2}\beta^2 + \frac{\alpha}{GF}\frac{dV}{dx}\beta\right] \qquad (22)$$

$$M_x = EJ\chi_x \qquad (23)$$

$$\gamma_x = \frac{\alpha}{GF}[0\ \ 0\ \ 0\ \ 0\ \ f_5\ \ f_6]X \qquad (24)$$

$$\varphi = \left[-\frac{dW}{dx}\beta + \frac{\alpha}{GF}V\right] \qquad (25)$$

In the above formulas $\beta=2/\Delta x$ is the factor that returns the two-unit length of the element to its true length.

b. Element stiffness matrix

Knowing the deflection function, the shear force function of the element, it is easy to calculate the element stiffness matrix. According to the Gaussian extremum principle method, we write the coercive quantity for the static problem as follows.

$$Z = \int_{-1}^{1} M_x[\chi_x]dx + \int_{-1}^{1} V[\gamma_x]dx \to Min \qquad (26)$$

χ_x and γ_x are expressions containing the unknowns X(i), so the stationary condition of (26) is rewritten as follows.

$$\delta Z = \int_{-1}^{1} M_x\delta[\chi_x]dx + \int_{-1}^{1} V\delta[\gamma_x]dx = 0 \qquad \text{or is}$$

$$[A_e] = \delta Z = \frac{\Delta x}{2}\begin{pmatrix} \int_{-1}^{1} M_x\left[\dfrac{\partial\chi_x}{\partial X(i)}\right]dx \\ + \int_{-1}^{1} V\left[\dfrac{\partial\gamma_x}{\partial X(i)}\right]dx \end{pmatrix} = 0 \qquad (27)$$

X(i) with (i=1÷6) are the hidden displacements, rotation angles and shear forces (W_1, W_2, θ_1, θ_2, Q_1, Q_2) at the two ends of the element, respectively, according to (20) rewritten as follows: $\{X_e\} = [W_1\ \ W_2\ \ \theta_1\ \ \theta_2\ \ Q_1\ \ Q_2]^T$

The factor $\Delta x/2$ to bring the integral from (-1) to (1) to the integral in terms of element length. For each (i) we get a row of 6 columns, in turn let i run from 1 to 6 and calculate (27) we get an element stiffness matrix [ae] of size (6x6), as follows:

$$[ae] =$$

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \frac{12EJ}{L^3} & -\frac{12EJ}{L^3} & \frac{6EJ}{L^2} & \frac{6EJ}{L^2} & 0 & 0 & 1 \\ & -\frac{12EJ}{L^3} & \frac{12EJ}{L^3} & -\frac{6EJ}{L^2} & -\frac{6EJ}{L^2} & 0 & 0 & 2 \\ & \frac{6EJ}{L^2} & -\frac{6EJ}{L^2} & \frac{4EJ}{L} & \frac{2EJ}{L} & -\begin{pmatrix}0.2*\\10^{-4}\end{pmatrix}L & \begin{pmatrix}0.2*\\10^{-4}\end{pmatrix}L & 3 \\ & \frac{6EJ}{L^2} & -\frac{6EJ}{L^2} & \frac{2EJ}{L} & \frac{4EJ}{L} & \begin{pmatrix}0.2*\\10^{-4}\end{pmatrix}L & -\begin{pmatrix}0.2*\\10^{-4}\end{pmatrix}L & 4 \\ & 0 & 0 & \begin{pmatrix}0.2*\\10^{-4}\end{pmatrix}L & -\begin{pmatrix}0.2*\\10^{-4}\end{pmatrix}L & \begin{pmatrix}0.667\\*10^{-5}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}0.333\\*10^{-5}\end{pmatrix}\frac{L^3}{EJ} & 5 \\ & 0 & 0 & -\begin{pmatrix}0.2*\\10^{-4}\end{pmatrix}L & \begin{pmatrix}0.2*\\10^{-4}\end{pmatrix}L & \begin{pmatrix}0.333\\*10^{-5}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}0.667\\*10^{-5}\end{pmatrix}\frac{L^3}{EJ} & 6 \end{matrix} \qquad (28)$$

The integrals in (19) can be calculated exactly or in terms of Gaussian approximate integrals (numerical integrals). After calculation, get matrix [ae](6x6) by (28).

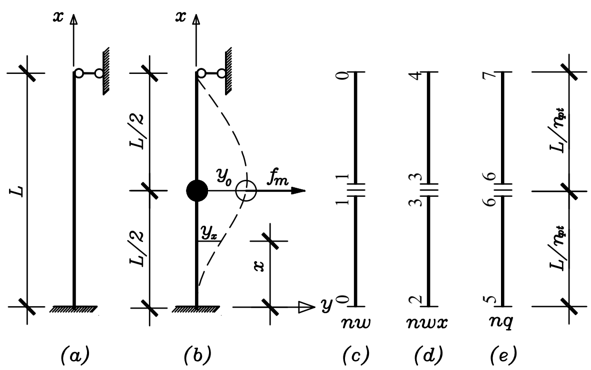
The matrix [ae] is called the element stiffness matrix, L is the length of an element. Because the deflection function of the element is a cubic polynomial, the forces acting and the inertia forces of the elements must all be distributed about its node. There are six unknowns we get six equations and have the following form.

$$[A_e]\{X_e\} = \{B_e\} \qquad (29)$$

where: $\{B_e\}$ is the element node force vector (for static problems), if at node (1) there is a force P, then the right side B(1)=P..., $\{B_e\}$ ={0} is vector "0" (for free vibration problem), if mass m is located at node (1) (displacement element) and inertia force $f_m=m\omega^2W_1$ then this component is included matrix [ae] at the following position: $ae(1,1)=m\omega^2$. Usually, we will include the inertial forces into the overall matrix of the bar. Knowing the element stiffness matrix, it is easy to construct the overall stiffness matrix of the bar. Assuming the bar has only one element, the matrix [ae] is the overall stiffness matrix of the bar. Assuming the displacement at node (1) is zero, then we drop row 1 column 1 of matrix [ae], assuming shear force $Q_2=0$ then we drop row 6 column 6 of [ae] because we don't have two this hidden.

4.2. Calculation examples

Example 1. The bar with one end fixed and the other pinned: Give the bar with one end fixed and the other pinned, length L, with mass evenly distributed over the length of the bar, bar with flexural stiffness EJ=const, figure 4a. Determine the natural frequency of the oscillation and the natural form of the bar.



Hình 4. The bar with one end fixed and the other pinned

Divide the bar into 2 elements (npt=2), the length of each element is Δx=L/2. Let nw, nwx, nq be the hidden numbers of displacement, rotation angle and shear force at the two ends of each element, respectively, and proceed to number the hidden numbers as shown in Figure 4, c, d, e.

$$\left.\begin{aligned} nw &= \begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix} \\ nwx &= \begin{bmatrix}2 & 3 & 3 & 4\end{bmatrix} \\ nq &= \begin{bmatrix}5 & 6 & 6 & 7\end{bmatrix}\end{aligned}\right\} \qquad \text{(a1)}$$

We have the general element stiffness matrix [ae] as follows:

[ae]=

1 2 3 5 6

$$\begin{bmatrix} \frac{96EJ}{L^3} & -\frac{96EJ}{L^3} & \frac{24EJ}{L^2} & \frac{24EJ}{L^2} & 0 & 0 \\ -\frac{96EJ}{L^3} & \frac{96EJ}{L^3} & -\frac{24EJ}{L^2} & -\frac{24EJ}{L^2} & 0 & 0 \\ \frac{24EJ}{L^2} & -\frac{24EJ}{L^2} & \frac{8EJ}{L} & \frac{4EJ}{L} & \frac{-L}{25000} & \frac{L}{25000} \\ \frac{24EJ}{L^2} & -\frac{24EJ}{L^2} & \frac{4EJ}{L} & \frac{8EJ}{L} & \frac{L}{25000} & \frac{-L}{25000} \\ 0 & 0 & \frac{-L}{25000} & \frac{L}{25000} & \begin{pmatrix}3.3341\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}1.6659\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} \\ 0 & 0 & \frac{L}{25000} & \frac{-L}{25000} & \begin{pmatrix}1.6659\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}3.3341\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} \end{bmatrix} \begin{matrix}1\\2\\3\\4\\5\\6\end{matrix} \qquad \text{(b1)}$$

According to (a) we see that element 1 has a displacement equals zero at node (1) nw(1)=0, so from the element matrix [ae] we delete row 1 column 1, the rest is that element stiffness 1, as follows:

[ae1]=

1 2 3 5 6

$$\begin{bmatrix} \frac{96EJ}{L^3} & -\frac{24EJ}{L^2} & -\frac{24EJ}{L^2} & 0 & 0 \\ -\frac{24EJ}{L^2} & \frac{8EJ}{L} & \frac{4EJ}{L} & \frac{-L}{25000} & \frac{L}{25000} \\ -\frac{24EJ}{L^2} & \frac{4EJ}{L} & \frac{8EJ}{L} & \frac{L}{25000} & \frac{-L}{25000} \\ 0 & \frac{-L}{25000} & \frac{L}{25000} & \begin{pmatrix}3.3341\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}1.6659\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} \\ 0 & \frac{L}{25000} & \frac{-L}{25000} & \begin{pmatrix}1.6659\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}3.3341\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} \end{bmatrix} \begin{matrix}1\\2\\3\\5\\6\end{matrix} \qquad \text{(c1)}$$

According to (a) we see that element 2 has a displacement equals zero at node (2) nw(2)=0, so from the element matrix [ae] we delete row 2 column 2, the rest is that element stiffness 2, as follows:

[ae2]=

1 3 4 6 7

$$\begin{bmatrix} \frac{96EJ}{L^3} & \frac{24EJ}{L^2} & \frac{24EJ}{L^2} & 0 & 0 \\ \frac{24EJ}{L^2} & \frac{8EJ}{L} & \frac{4EJ}{L} & \frac{-L}{25000} & \frac{L}{25000} \\ \frac{24EJ}{L^2} & \frac{4EJ}{L} & \frac{8EJ}{L} & \frac{L}{25000} & \frac{-L}{25000} \\ 0 & \frac{-L}{25000} & \frac{L}{25000} & \begin{pmatrix}3.3341\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}1.6659\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} \\ 0 & \frac{L}{25000} & \frac{-L}{25000} & \begin{pmatrix}1.6659\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}3.3341\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} \end{bmatrix} \begin{matrix}1\\3\\4\\6\\7\end{matrix} \qquad \text{(d1)}$$

Create a matrix "0" of size equal to the total number of unknowns of the problem (7 unknowns), so we have an overall "0" matrix [A(0)] of size (7x7), then assemble [ae1] and [ae2] into [A(0)], the terms of the same address (i,j) are added, finally we get the overall stiffness matrix [A] as follows:

[A]=

1 2 3 4 5 6 7

$$\begin{bmatrix} \frac{192EJ}{L^3} & -\frac{24EJ}{L^2} & 0 & \frac{24EJ}{L^2} & 0 & 0 & 0 \\ -\frac{24EJ}{L^2} & \frac{8EJ}{L} & \frac{4EJ}{L} & 0 & \frac{-L}{25000} & \frac{L}{25000} & 0 \\ 0 & \frac{4EJ}{L} & \frac{16EJ}{L} & \frac{4EJ}{L} & \frac{L}{25000} & \frac{-2L}{25000} & \frac{L}{25000} \\ \frac{24EJ}{L^2} & 0 & \frac{4EJ}{L} & \frac{8EJ}{L} & 0 & \frac{L}{25000} & \frac{-L}{25000} \\ 0 & \frac{-L}{25000} & \frac{L}{25000} & 0 & \begin{pmatrix}3.3341\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}1.6659\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & 0 \\ 0 & \frac{L}{25000} & \frac{-2L}{25000} & \frac{L}{25000} & \begin{pmatrix}1.6659\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}6.6682\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}1.6659\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} \\ 0 & 0 & \frac{L}{25000} & \frac{-L}{25000} & 0 & \begin{pmatrix}1.6659\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}3.3341\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} \end{bmatrix} \qquad \text{(e1)}$$

Note that in addition to the hidden displacements, rotation angles, and shear forces of the bar, we must also consider the unknowns that are the Lagrange λ factors of the constrain conditions at the two ends of the bar.

In this lesson, we also add three unknowns λ_1, λ_2 and λ_3 which are three Lagrange factors corresponding to three constraints: the rotation angle at the bar foot is zero, the bending moment at the bar end is zero and the forced displacement at the end of element 1 of the bar equals y0. As follows:

$$g_1 = \varphi = \lambda_1\left[-\frac{dW}{dx}\beta + \frac{\alpha}{GF}V\right]_{[\text{phantu}(1)\ \text{tai}\ x=-1]} = 0$$

$$g_2 = \chi_x = \lambda_2\left[-\frac{d^2W}{dx^2}\beta^2 + \frac{\alpha}{GF}\frac{dV}{dx}\beta\right]_{[\text{phantu}(1)\ \text{tai}\ x=-1]} = 0$$

$$g_3 = \lambda_3\left[y(\text{phantul},2) - y0\right] = 0 \qquad \text{(f1)}$$

Thus, the overall stiffness matrix [A] will be expanded by three rows, three columns become [A(10X10)], not shown here because of its large size. In the final overall matrix, also consider the inertia force with specific values and positions as follows: Because the bar is divided into 2 elements, the two ends of the bar are fixed, so there is only the end node of element 1 or the beginning node of the element 2 is the point where the force of inertia is applied with a value of:

$$f_m = -\frac{1}{2} EJk_1^2 L \quad \text{where: } k_1 = \frac{m\omega^2}{EJ}; \qquad \omega = k_1\sqrt{\frac{EJ}{m}} \quad \text{(g1)}$$

Thus, after expanding by three rows and three columns, we get the final overall stiffness matrix of size [A] (10x10), corresponding to ten equations of the form:

$$[A]\{X\} = \{B\} \quad \text{(h1)}$$

Where: {X} is the hidden vector and {B} is the node force vector, {B} is the column vector of size (10x1), all terms of the vector {B} are zero, except B(10,1)=y0.

$$\{X\} = \begin{Bmatrix} W_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{matrix} \qquad \{B\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ y0 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{matrix}$$

Solving the system of equations (h1), we get the equation λ_3 (unknown number 10, due to the forced displacement at the top of the bar equal to y0, according to (d)), the equation λ_3 has the following form:

h/L=1/100 (not considering the lateral shear strain)

$$\lambda_3 = 0.45686x10^{-9} EJ.y0\left[\begin{matrix}(-240086400000 \\ +1094425081.k_1^2.L^4)\end{matrix} \Big/ L^3\right]$$

solving the equation $\lambda_3=0$ we get: k1= 14.81/L2

replacing k1 in (g1), we have:

$$\omega = 14{,}81\sqrt{\frac{EJ}{mL^4}}; \qquad \text{Analytic solution: } \omega = 15{,}41\sqrt{\frac{EJ}{mL^4}}$$

h/L=1/3 (considering the lateral shear strain)

$$\lambda_3 = 0.16077x10^{-2} EJ.y0\left[\begin{matrix}(-53760 \\ +311.k_1^2.L^4)\end{matrix} \Big/ L^3\right]$$

Solving the equation $\lambda_3=0$ we get: $k_1= 13.15/L^2$

replacing k1 in (e), we have: $\omega = 13{,}15\sqrt{\frac{EJ}{mL^4}}$

Comment: Because of dividing the bar into two elements, on the bar there is an inertial force concentrated in the middle of the bar (system of one degrees of freedom), so only one fundamental frequency is obtained with an error of 3.89% compared to the exact result, to get asymptotic results with exact results, we need to discretize the bar into more elements, for example, divide the bar into 8 elements, we get the result in two cases not considering (h/L) =1/100) and considering (h/L=1/10; and h/L=1/3) on the effect of lateral shear strain, as follows:

h/L=1/100 (not cosidering the lateral shear strain)

When dividing the bar into 8 elements, the problem will have a total of 28 unknowns, including (7 hidden displacements, 9 hidden angles of rotation, 9 hidden forces of shear and three Lagrange factors λ_1, λ_2 and λ_3 respectively corresponding to three constraints, the rotation angle at the bar foot clamp is zero, the moment at the bar end is zero and the forced displacement at the bar end is equal to y0), the overall stiffness matrix [A](28X28), So we get 28 equations of the form (f):

$$[A]\{X\} = \{B\}$$

Solving this equation, we get λ_3 of the following form:

Case 1: h/L=1/100 (without considering the lateral shear strain), we have:

λ_3=.125xEJxy0/l^3(-.82699x$10^{59}k_1^4l^8$+.10015 x$10^{56}k_1^6l^{12}$+.16364 x$10^{63}k_1^2l^4$-.48687 x$10^{41}k_1^{12}l^{24}$+

.69477 x$10^{46}k_1^{10}l^{20}$-.41469 x$10^{51}k_1^8l^{16}$+.11854 x$10^{36}k_1^{14}l^{28}$

-.34296 x10^{65}) (i1)

Case 2: h/L=1/3 (considering the shear strain)

λ_3=.125/l^3EJ.y0(.44457x$10^{41}k_1^2l^4$+3916120115755 1431.$k_1^{14}l^{28}$-.64559 x$10^{38}k_1$^$^4l^8$-6666481932951165665280.$k_1^{12}l$^^{24}

-.40896 x10^{43}+.277492 x$10^{35}k_1^6l^{12}$-

4492738334263726156535562240000.$k_1^8l^{16}$+28821139213949 7256570060800.$l^{20}k_1^{10}$) (k1)

Solving equations (i1) and (k1) we get the first line and the third row of table 1. We see that λ_3 is a 14 degree polynomial of k1, so solving $\lambda_3=0$ we get 14 eigen frequencies ωi of the problem corresponds to two cases h/L=1/100 and h/L=1/3, here only the first 3 frequencies are given (table 1) along with three types of oscillations and three shape of the corresponding shear force line, figure 5, 6.

Table 1. The natural frequency of oscillation of the bar with one end fixed and the other pinned calculated for the two cases h/l. Split bar by 8 elements

Rate h/l	**The first three frequencies** $\omega_i = k_{1i}\sqrt{\frac{EJ}{mL^4}}$		
	k_{11}	k_{12}	k_{13}
1/100	15.401	49.874	103.759
1/10	14.123	45.313	91.965
1/3	10.402	31.035	55.142

Table 2. Comparison of the natural frequency of oscillations of the bar with one end fixed and the other pinned in the two cases without considering and with considering lateral strain.

Cases	**The first three frequencies** $\omega_i = k_{1i}\sqrt{\frac{EJ}{mL^4}}$		
	k_{11}	k_{12}	k_{13}
Not considered	15.401	49.874	103.759
Considered	10.153	30.571	54.098
Difference (%)	34.075	38.707	47.861

Table 3. Comparison of the natural frequency of oscillations of the bar with one end fixed and the other pinned determined according to the finite element and the exact results:

Cases	The first three frequencies $\omega_i = k_{1i}\sqrt{\frac{EJ}{mL^4}}$		
	k_{11}	k_{12}	k_{13}
Analytical method	15.418	49.964	104.266
Finite element method	15.401	49.874	103.759
Difference (%)	0.11	0.18	0.48

Comment:

- According to Tables 1 and 2, we can see that when considering the lateral shear strain, the vibration frequency of the bar is greatly reduced, the first frequency decreases by 34.075%, the second frequency decreases by 38.707%, and the third frequency decreases by 47.861%.

- We see, just discretizing the bar into two elements, we have obtained results very close to the results found by analytical methods (error 3.89%), when dividing the bar into eight elements, the result received has an error of close to zero, the error is (0.11% for the first fundamental frequency, 0.18% for the second and 0.48% for the third frequency). Indeed , the natural frequency k11=15.401/L2 (number of elements equals 3) is approximately the same as the analytical result.

With the oscillation frequencies received above, we have the corresponding vibration patterns and shear force lines, below the author presents three types of vibration and three types of shear force lines corresponding to the first three frequencies of vibration, figure 5, 6.

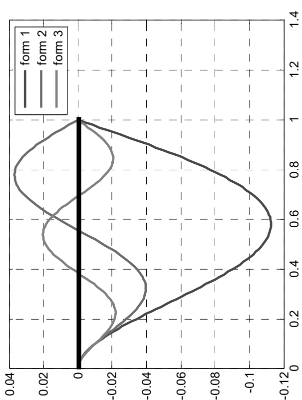


Figure 5. Three types of oscillations corresponding to the first three frequencies

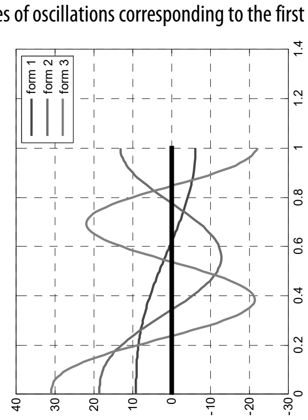


Figure 6. Three types of shear lines corresponding to the first three frequencies

Example 2. The bar with hinged ends

Given a straight bar with two joint ends, of length L, with a mass evenly distributed throughout the length of the bar, the bar has flexural stiffness EJ=const, figure 18a. Determine the natural frequency of the oscillation and the natural form of the bar.

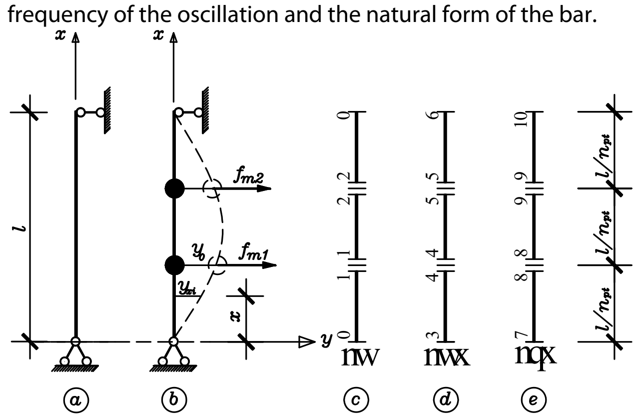


Figure 7. The bar with hinged ends

Divide the bar into 3 elements (npt=3), the length of each element is Δx=L/3. Let nw, nwx, nq be the hidden numbers of displacement, rotation angle and shear force at the ends of each element, respectively, and proceed to number the hidden numbers as shown in Figure 18, c, d, e.

$$\left.\begin{array}{l} nw = [0 \quad 1 \quad 1 \quad 2 \quad 2 \quad 0] \\ nwx = [3 \quad 4 \quad 4 \quad 5 \quad 5 \quad 6] \\ nqx = [7 \quad 8 \quad 8 \quad 9 \quad 9 \quad 10] \end{array}\right\} \quad (a2)$$

We have the general element stiffness matrix [ae]:

[ae]=

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{324EJ}{L^3} & -\frac{324EJ}{L^3} & \frac{54EJ}{L^2} & \frac{54EJ}{L^2} & 0 & 0 \\ -\frac{324EJ}{L^3} & \frac{324EJ}{L^3} & -\frac{54EJ}{L^2} & -\frac{54EJ}{L^2} & 0 & 0 \\ \frac{54EJ}{L^2} & -\frac{54EJ}{L^2} & \frac{12EJ}{L} & \frac{6EJ}{L} & \frac{-3L}{50000} & \frac{3L}{50000} \\ \frac{54EJ}{L^2} & -\frac{54EJ}{L^2} & \frac{6EJ}{L} & \frac{12EJ}{L} & \frac{3L}{50000} & \frac{-3L}{50000} \\ 0 & 0 & \frac{-3L}{50000} & \frac{3L}{50000} & \begin{pmatrix} 2.2234 \\ *10^{-6} \end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix} 1.1099 \\ *10^{-6} \end{pmatrix}\frac{L^3}{EJ} \\ 0 & 0 & \frac{3L}{50000} & \frac{-3L}{50000} & \begin{pmatrix} 1.1099 \\ *10^{-6} \end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix} 2.2234 \\ *10^{-6} \end{pmatrix}\frac{L^3}{EJ} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array} \quad (b2)$$

According to (a) we see that element 1 has a displacement equal zero at node (1) nw(1)=0, so from the element matrix [ae] we delete row 1 column 1, the rest is that element stiffness 1, as follows:

[ae1]=

$$\begin{array}{c} \begin{array}{ccccc} 1 & 3 & 4 & 7 & 8 \end{array} \\ \begin{bmatrix} \frac{324EJ}{L^3} & -\frac{54EJ}{L^2} & -\frac{54EJ}{L^2} & 0 & 0 \\ -\frac{54EJ}{L^2} & \frac{12EJ}{L} & \frac{6EJ}{L} & \frac{-3L}{50000} & \frac{3L}{50000} \\ -\frac{54EJ}{L^2} & \frac{6EJ}{L} & \frac{12EJ}{L} & \frac{3L}{50000} & \frac{-3L}{50000} \\ 0 & \frac{-3L}{50000} & \frac{3L}{50000} & \begin{pmatrix} 2.2234 \\ *10^{-6} \end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix} 1.1099 \\ *10^{-6} \end{pmatrix}\frac{L^3}{EJ} \\ 0 & \frac{3L}{50000} & \frac{-3L}{50000} & \begin{pmatrix} 1.1099 \\ *10^{-6} \end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix} 2.2234 \\ *10^{-6} \end{pmatrix}\frac{L^3}{EJ} \end{bmatrix} \begin{array}{c} 1 \\ 3 \\ 4 \\ 7 \\ 8 \end{array} \end{array} \quad (c2)$$

According to (a), we see that element 2 is the middle element (not related to boundary conditions), so the common element matrix [ae] is the element 2's stiffness matrix, as follows:

[ae2]=

$$
\begin{array}{c}
\begin{matrix} 1 & 2 & 4 & 5 & 8 & 9 \end{matrix}\\
\left[\begin{matrix}
\frac{324EJ}{L^3} & -\frac{324EJ}{L^3} & \frac{54EJ}{L^2} & \frac{54EJ}{L^2} & 0 & 0\\
-\frac{324EJ}{L^3} & \frac{324EJ}{L^3} & -\frac{54EJ}{L^2} & -\frac{54EJ}{L^2} & 0 & 0\\
\frac{54EJ}{L^2} & -\frac{54EJ}{L^2} & \frac{12EJ}{L} & \frac{6EJ}{L} & \frac{-3L}{50000} & \frac{3L}{50000}\\
\frac{54EJ}{L^2} & -\frac{54EJ}{L^2} & \frac{6EJ}{L} & \frac{12EJ}{L} & \frac{3L}{50000} & \frac{-3L}{50000}\\
0 & 0 & \frac{-3L}{50000} & \frac{3L}{50000} & \begin{pmatrix}2.2234\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}1.1099\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ}\\
0 & 0 & \frac{3L}{50000} & \frac{-3L}{50000} & \begin{pmatrix}1.1099\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}2.2234\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ}
\end{matrix}\right]
\begin{matrix} 1\\ 2\\ 4\\ 5\\ 8\\ 9 \end{matrix}
\end{array}
\qquad \text{(d2)}
$$

According to (a) we see that element 3 has a displacement equal zero at node (2) nw(2)=0, so from the element matrix [ae] we delete row 2 column 2, the rest is matrix element stiffness 3, as follows:

[ae3]=

$$
\begin{array}{c}
\begin{matrix} 2 & 5 & 6 & 9 & 10 \end{matrix}\\
\left[\begin{matrix}
\frac{324EJ}{L^3} & \frac{54EJ}{L^2} & \frac{54EJ}{L^2} & 0 & 0\\
\frac{54EJ}{L^2} & \frac{12EJ}{L} & \frac{6EJ}{L} & \frac{-3L}{50000} & \frac{3L}{50000}\\
\frac{54EJ}{L^2} & \frac{6EJ}{L} & \frac{12EJ}{L} & \frac{3L}{50000} & \frac{-3L}{50000}\\
0 & \frac{-3L}{50000} & \frac{3L}{50000} & \begin{pmatrix}2.2234\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}1.1099\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ}\\
0 & \frac{3L}{50000} & \frac{-3L}{50000} & \begin{pmatrix}1.1099\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ} & \begin{pmatrix}2.2234\\ *10^{-6}\end{pmatrix}\frac{L^3}{EJ}
\end{matrix}\right]
\begin{matrix} 2\\ 5\\ 6\\ 9\\ 10 \end{matrix}
\end{array}
\qquad \text{(e2)}
$$

Create a matrix “0” of size equal to the total number of unknowns of the problem (10 unknowns), so we have an overall “0” matrix [A(0)] size (10x10), then assemble [ae1], [ae2] and [ae3] into [A(0)], the terms of the same address (i,j) are added, finally we get the overall stiffness matrix [A] (not shown in here because the size is too large).

Note that in addition to the hidden displacements, rotation angles, and shear forces of the bar, we must also consider the unknowns that are the Lagrange λ factors of the constrain conditions at the two ends of the bar.

In this lesson, we also add three unknowns λ_1, λ_2 and λ_3 which are three Lagrange factors corresponding to three constraints: the moment at the ends of the bar is zero and the forced displacement at the end of element 1 of the bar is equal to y0. As follows:

$$g_1=\lambda_1\left[-\frac{d^2W}{dx^2}\beta^2+\frac{\alpha}{GF}\frac{dV}{dx}\beta\right]_{[\text{phantu}(1)\ \text{tai}\ x=-1]}=0$$

$$g_2=\lambda_2\left[-\frac{d^2W}{dx^2}\beta^2+\frac{\alpha}{GF}\frac{dV}{dx}\beta\right]_{[\text{phantu}(3)\ \text{tai}\ x=1]}=0$$

$$g_3=\lambda_3\left[y(\text{phantul},2)-y0\right]=0 \qquad \text{(f2)}$$

Thus, the overall stiffness matrix [A] will be expanded by three rows and three columns to become [A(13X13)], not shown here because of its large size.

In the final overall matrix, also consider the inertial force with specific values and positions as follows:

Because the bar is divided into 3 elements, the two ends of the bar are fixed, so only the end node of element 1 and the end node of element 2 is the point where the inertia force is applied with the value of:

$$f_{m1}=f_{m2}=-\frac{1}{2}EJk_1^2L\,,$$

$$\text{here: } k_1=\frac{m\omega^2}{EJ};\qquad \omega=k_1\sqrt{\frac{EJ}{m}} \qquad \text{(g2)}$$

Thus, after expanding by three rows and three columns, we get the final overall stiffness matrix of [A] size (13x13), corresponding to 13 equations of the form:

$$[A]\{X\}=\{B\} \qquad \text{(h2)}$$

Where: $\{X\}$ is the hidden vector and the node force vector {B} is a column vector of size (13x1), all terms in vector {B} are zero, except B(13 ,1)=y0.

$$
\{X\}=\left\{\begin{matrix} W_1\\ W_2\\ \theta_3\\ \theta_4\\ \theta_5\\ \theta_6\\ Q_7\\ Q_8\\ Q_9\\ Q_{10}\\ \lambda_1\\ \lambda_2\\ \lambda_3 \end{matrix}\right\}
\begin{matrix} 1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ 8\\ 9\\ 10\\ 11\\ 12\\ 13 \end{matrix}
\qquad
\{B\}=\left\{\begin{matrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ y0 \end{matrix}\right\}
\begin{matrix} 1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ 8\\ 9\\ 10\\ 11\\ 12\\ 13 \end{matrix}
$$

Solving the system of equations (e), we get the equation λ_3 (unknown number 13, due to the forced displacement at the top of the bar equal to y0, according to (c)), the equation λ_3 has the following form:

h/L=1/100 (without considering the lateral shear strain)

$$\lambda_3=0.33333xEJ.y0\left[\begin{matrix}(8857350000000000.\\ -97220995200000.k_1^2.L^4\\ +62545906561.k_1^2.L^4)\end{matrix}\Big/ L^8\right]$$

Solving the equation λ_3=0 we get:

k11= 9.858/L2; k12= 38.173/L2;

Substituting k1 into (g2), we have:

$$\omega 1=9.858\sqrt{\frac{EJ}{mL^4}}\,;\qquad \omega 2=38.173\sqrt{\frac{EJ}{mL^4}}$$

h/L=1/3 (taking into account the shear strain)

$$\lambda_3=0.83333x10^{-1}EJ.y0\left[\begin{matrix}(-301320.k_1^2.L^4\\ +304.k_1^4.L^8+22143375)\end{matrix}\Big/ L^3\right]$$

Solving the equation λ_3=0 we get:k_{11}= 8.94/L^2 ; k_{11}= 30.186/L^2, replacing k1 in (34), we have:

$$\omega l = 8.94\sqrt{\frac{EJ}{mL^4}}\ ;\ \omega l = 30.186\sqrt{\frac{EJ}{mL^4}}$$

Comment: Because of dividing the bar into three elements, on the bar there are two inertial forces concentrated at the end of element 1 and the end of element 3 (two degrees of freedom), so only two fundamental frequencies are obtained with an error of 0.11 % compared to the exact result, to get the result asymptotic to the exact result, we need to discretize the bar into more elements, for example, divide the bar into 8 elements, we get the result in two cases. consider (h/L=1/100) and take into account (h/L=1/10; and h/L=1/3) on the effect of lateral shear strain, as follows:

When dividing the bar into 6 elements, the problem will have a total of 22 unknowns, including (5 displacements, 7 rotations, 7 shear forces and three Lagrange factors λ_1, λ_2 and λ_3, respectively. corresponding to the three constraints, the moment at the ends of the bar is zero and the forced displacement at the end of the element 1= y0), the overall stiffness matrix [A](22X22), So we get 22 equations with form (h2):

$$[A]\{X\}=\{B\}$$

Solving this equation, we get λ_3 of the following form:

Case 1: h/L=1/100 (without considering the lateral shear strain), we have:

λ3=.16666*EJ*y0*(.75155e41*k1^2*l^4-.59016e38*k1^4*l^8-.676740e43+.92947e34*k1^6*l^12-41709450678164879856645600000.*k1^8*l^16+5188559845051279738284401.*k1^10*l^20) (i2)

Case 2: h/L=1/3 (with considering the lateral shear strain), we have:

λ3=.16666xEJ.y0(342733447.k110l20-0436948463600.k18l16 +81544352389632000.k16l^12-196698479456231424000.k14l8+123626142353654415360000.k12*l4-866235338411920588800000.) (k2)

Solving equations (i2) and (k2) we get the first line and the third row of table 4. We see that λ_3 is a 10th order polynomial of k1, so solving λ_3=0 we get 10 natural frequencies. ωi of the problem corresponds to the two cases h/L=1/100 and h/L=1/3, here only the first 3 frequencies are given (table 4) along with three types of oscillations and three shape of the corresponding shear force line, figure 8, 9.

Table 4. The natural frequency of oscillations of the bar with hinged ends calculated for the two cases h/l. Split bar by 6 elements

Rate h/l	The first three frequencies $\omega_i = k_{1i}\sqrt{\frac{EJ}{mL^4}}$		
	k11	k12	k13
1/100	9.868	39.450	88.586
1/10	9.773	38.001	81.857
1/5	9.501	34.424	68.146
1/3	8.938	28.835	76.494

Table 5. Comparison of the natural frequency of oscillations of the bar with hinged ends because in the two cases, with and without considering to the lateral shear strain.

case	The first three frequencies $\omega_i = k_{1i}\sqrt{\frac{EJ}{mL^4}}$		
	k_{11}	k_{12}	k_{13}
Not considered	9.868	39.420	88.110
Considered	8.938	28.884	82.832
Difference (%)	9.42	26.72	5.99

Table 6. Comparison of natural oscillation frequencies of the bar with hinged ends determined according to the finite element method and exact results:

Cases	the first three frequencies $\omega_i = k_{1i}\sqrt{\frac{EJ}{mL^4}}$		
	k_{11}	k_{12}	k_{13}
Analytical method	9.869	39.478	88.830
Finite element method	9.868	39.420	88.110
Difference (%)	0.01	0.14	0.81

Comment:

- According to Table 4, 5, we can see that, when considering the lateral shear strain, the vibration frequency of the bar is relatively large, the first frequency is reduced by 9.42%, the second frequency is reduced by 26.72%, and the third frequency is reduced by 5.99%. .

- We see, just discretizing the bar into three elements has obtained results very close to the results found by the analytical method (error 0.11%), when dividing the bar into 6 elements, the result received has an error of close to zero, the error is (0.01% for the first fundamental frequency, 0.14% for the second and 0.81% for the third frequency). Indeed , the natural frequency k11=9.868/L2 (number of elements is 6) coincides with the analytical result.

With the oscillation frequencies received above, we have the corresponding vibration patterns and shear force lines, below the author presents three types of vibration and three types of shear force lines corresponding to the first three frequencies of vibration. first, figure 8, 9.

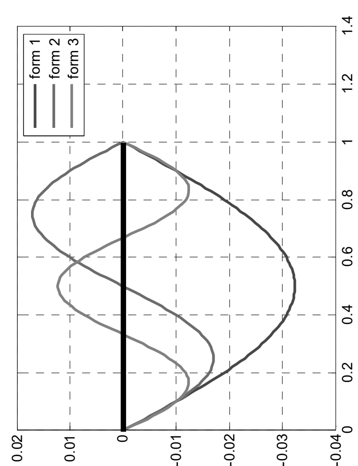


Figure 8. Three types of oscillations corresponding to the first three frequencies

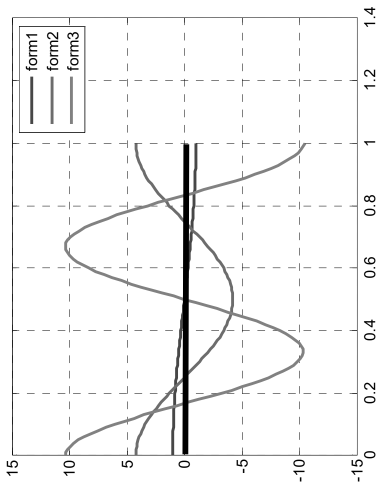


Figure 9. Three types of shear lines corresponding to the first three frequencies

5. CONCLUSIONS

With the combination of forced displacement method and finite element method, the author has successfully built the problem of free oscillation of the bar taking into account the influence of lateral shear deformation, finding a numerical solution of the problem. The problems are completely consistent with the results of solving by existing methods. When we divide the bar into many elements, we will get many exact solutions. For the bar with one end fixed and the other pinned just need to divide the bar into 8 elements, the results are almost identical to the results when solving by analytical method, the error is negligible (the error is 0.11 respectively). %, 0.18% and 0.48% for the first three frequencies - table 3).

The oscillation frequencies obtained by the finite element method almost coincide with the results obtained by the analytical method in the case that the effect of the lateral shear strain (h/L=1/100) is not taken into account, the error is considered as zero, which proves the reliability and efficiency of the finite element method for bar vibration problems.

The results are obtained in two cases with and without taking into account the influence of the lateral shear deformation of large changes (the natural frequency decreases by 34.075%, 38.707% and 47.861%, respectively, corresponding to the first three vibration frequencies. - Table 2) for the head-mount bar - the joint head, and for the double-ended bar, the frequency reduction is 31.17%, 45.82. The frequency of oscillation changes depends on the ratio h/L, the larger the h/L, the more the frequency decreases (Tables 1, 4). This shows that it is necessary to consider the effect of lateral shear strain when ($h/L \geq 1/10$).

When not considering the lateral shear strain ($G \to \infty$) or ($h \to 0$) the expressions, the stiffness matrix and the obtained results coincide with the problem built according to the traditional Euler - Bernoulli theory.

When using forced displacement method to solve the problem of free oscillation of the bar, it immediately gives us the polynomial equation that determines the natural frequency of the bar without having to go through complicated transformations to bring the matrix back to diagonal matrix and no need to look up the table. The finite element method combined with the forced displacement method presented here gives us a very efficient algorithm, a new approach to evaluate the oscillation frequency of the eigenvalue problem of bars and systems of bars. That may be the most prominent advantage of this article.

Recommendation: Use the new approach developed above to find eigenvalues and eigenvectors of mechanical problems in particular and find solutions of problems with the right side equal to zero in general.

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