BOUNDS OF p-ADIC WEIGHTED BILINEAR HARDY-CESÀRO OPERATORS 66ON PRODUCT OF LEBESGUE SPACES

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Abstract: *In this paper we aim to investigate the boundedness of* $U_{\psi,\vec{s}}^{p,2,n}$ *on the product of* p*-adic weighted Lebesgue spaces. We obtain the necessary and sufficient conditions on weight functions to ensure the boundedness of that operator on the product of* p*-adic weighted Lebesgue spaces. Moreover, we obtain the corresponding operator norms.*

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1. INTRODUCTION

Theories of functions from $\mathbb{Q}_p^n$ into $\mathbb{C}$ play an important role in the theory of the p-adic quantum mechanics, the theory of p-adic probability. As far as we know, the studies of the p-adic Hardy operators and p-adic Hausdorff operators are also useful for p-adic analysis [4,5,6,14,24,27,28].

The weighted Hardy averaging operators are defined for measurable functions on $\mathbb{Q}_p$ by:

$$U_\psi^p f(x) = \int_{\mathbb{Z}_p^\star} f(tx)\psi(t)dt, \quad x \in \mathbb{Q}_p^d, \qquad (1.1)$$

here $\mathbb{Z}_p^\star$ is the ring of p-adic non-zero integers, and dx is the Haar measure on $\mathbb{Q}_p$. Rim and Lee [24] considered the problem of characterizing function ψ on $\mathbb{Z}_p^\star$, so that we have inequalities:

$$\left\| U_\psi^p f \right\|_X \leq C \parallel f \parallel_X$$

where X is p-adic Lebesgue or BMO space. The corresponding best constants C are also obtained by these authors.

Hung [14] considered a more general class of p-adic weighted Hardy averaging operators, which are called p-adic Hardy-Cesaro operators, defined as:

$$U_{\psi,s}^{p} f(x) = \int_{\mathbb{Z}_p^*} f(s(t)x)\psi(t)dt, \qquad (1.2)$$

where $s: \mathbb{Z}_p^* \to \mathbb{Q}_p$ and $\psi: \mathbb{Z}_p^* \to [0; \infty)$ are measurable funtions.

The characterizations on funtion $\psi(t)$, under certain conditions on $s(t)$, so that:

$$\left\| U_{\psi,s}^{p} f \right\|_X \leq C \parallel f \parallel_X$$

for all $f \in X$, where X is p-adic Lebesgue space, are obtained. The best constants C in the above inequalities are worked out too. It is interesting to notice that, by applying the boundedness of $U_{\psi,s}$ on p-adic weighted Lebesgue spaces, Hung gives a relation between $p-$ adic Hardy operators and discrete Hardy inequalities on the real field.

In [15], Hung and Ky gave the definition of the weighted multilinear Hardy-Cesàro operators $U_{\psi,\vec{s}}^{m,n}$ to be:

Definition 1.1. Let $m, n \in \mathbb{N}, \psi: [0,1]^n \to [0, \infty), s_1, \dots, s_m: [0,1]^n \to \mathbb{R}$ be measurable funtions. The weighted multilinear Hardy-Cesàro operators $U_{\psi,\vec{s}}^{m,n}$ is defined by:

$$U_{\psi,\vec{s}}^{m,n}(\vec{f})(x) = \int_{[0,1]^n} \left(\prod_{k=1}^{n} f_k(s_k(t)x) \right) \psi(t)dt, \qquad (1.3)$$

where $\vec{f} = (f_1, \dots, f_m), \vec{s} = (s_1, \dots, s_m)$.

The authors obtain the sharp bounds of $U_{\psi,\vec{s}}^{m,n}$ on the product of Lebesgue spaces and central Morrey spaces. In our paper, we define the p-adic weighted bilinear Hardy-Cesàro operators $U_{\psi,\vec{s}}^{p,2,n}$ as follow:

Definition 1.2. Let n be positive interger numbers and $\psi: \left(\mathbb{Z}_p^*\right)^n \to [\ 0; \infty), \vec{s} = (s_1, s_2): \left(\mathbb{Z}_p^*\right)^n \to \mathbb{Q}_p^2$ be measurable. The p-adic weighted bilinear Hardy-Cesàro operators $U_{\psi,\vec{s}}^{p,2,n}$, which define on $\vec{f} = (f_1, f_2) : \mathbb{Q}_p^d \to \mathbb{C}^2$ vector of measurable funtions, by

$$U_{\psi,\vec{s}}^{p,2,n}(f_1, f_2)(x) = \int_{\left(\mathbb{Z}_p^*\right)^n} \left(\prod_{k=1}^{2} f_k(s_k(t)x) \right) \psi(t)dt,$$

Our paper is organized as follow. In Section 2 we give the content of this paper including the notation and definitions that we shall use in the sequel. We define the p-adic weighted Lebesgue spaces $L_\omega^q\left(\mathbb{Q}_p^d\right)$. We also state the main results on the boundedness of $U_{\psi,\vec{s}}^{p,2,n}$ on the p-adic weighted Lebesgue space and work out the norms of $U_{\psi,\vec{s}}^{p,2,n}$ on such space. In Section 3 we give the conclusion of this paper.

2. CONTENT

2.1. Basic notions and lemmas

Let p be a prime number and let $r \in \mathbb{Q}^\star$. Write $r = p^\gamma \frac{a}{b}$ where a and b are integers not divisible by p. Define the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$ by $|r|_p = p^{-\gamma}$ and $|0|_p = 0$. The absolute value $|\cdot|_p$ gives a metric on $\mathbb{Q}$ defined by $d_p(x,y) = |x-y|_p$. We denote by $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to the metric d. $\mathbb{Q}_p$ with natural operations and topology induced by the metric d_p is a locally compact, non-discrete, complete and totally disconnected field. A non-zero element x of $\mathbb{Q}_p$, is uniquely represented as a canonical form $x = p^\gamma(x_0 + x_1 p + x_2 p^2 + \cdots)$ where $x_j \in \mathbb{Z}/p\mathbb{Z}$ and $x_0 \neq 0$. We then have $|x|_p = p^{-\gamma}$. Define $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ and $\mathbb{Z}_p^\star = \mathbb{Z}_p \setminus \{0\}$.

$\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ contains all n-tuples of $\mathbb{Q}_p$. The norm on $\mathbb{Q}_p^n$ is $|x|_p = \max_{1\leq k\leq n} |x_k|_p$ for $x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$. The space $\mathbb{Q}_p^n$ is complete metric locally compact and totally disconnected space. For each $a \in \mathbb{Q}_p$ and $x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$, we denote $ax = (ax_1, \dots, ax_n)$. For $\gamma \in \mathbb{Z}$, we denote B_γ as a γ-ball of $\mathbb{Q}_p^n$ with center at 0 , containing all x with $|x|_p \leq p^\gamma$, and $S_\gamma = B_\gamma \setminus B_{\gamma-1}$ its boundary. Also, for $a \in \mathbb{Q}_p^d, B_\gamma(a)$ consists of all x with $x - a \in B_\gamma$, and $S_\gamma(a)$ consists of all x with $x - a \in S_\gamma$.

Since $\mathbb{Q}_p^d$ is a locally-compact commutative group with respect to addition, there exists the Haar measure dx on the additive group of $\mathbb{Q}_p^d$ normalized by $\int_{B_0} dx = 1$. Then $d(ax) = |a|_p^d dx$ for all $a \in \mathbb{Q}_p^\star$, $\left|B_\gamma(x)\right| = p^{d\gamma}$ and $\left|S_\gamma(x)\right| = p^{d\gamma}(1 - p^{-d})$.

We shall consider the class of weights $\mathcal{W}_\alpha$, which consists of all nonnegative locally integrable function ω on $\mathbb{Q}_p^d$ so that $\omega(tx) = |t|_p^\alpha \omega(x)$ for all $x \in \mathbb{Q}_p^d$ and $t \in \mathbb{Q}_p^\star$ and $0 < \int_{S_0} \omega(x)dx < \infty$. It is easy to see that $\omega(x) = |x|_p^\alpha$ is in $\mathcal{W}_\alpha$ if and only if $\alpha > -d$.

Definition 2.1. Let ω be any weight function on $\mathbb{Q}_p^d$, that is a nonnegative, locally integrable function from $\mathbb{Q}_p^d$ into $\mathbb{R}$. Let $1 \leq r < \infty$, the p-adic weighted Lebesgue spaces $L_\omega^r\left(\mathbb{Q}_p^d\right)$ be the space of complexvalued functions f on $\mathbb{Q}_p^d$ so that

$$\| f \|_{L_\omega^r(\mathbb{Q}_p^d)} = \left(\int_{\mathbb{Q}_p^d} |f(x)|^r \omega(x) dx \right)^{1/r} < \infty$$

For further readings on p-adic analysis, see [25,26]. Here, some often used computational principles are worth mentioning at the outset. First, if $f \in L_\omega^1\left(\mathbb{Q}_p\right)$ we can write

$$\int_{\mathbb{Q}_p^d} f(x)\omega(x)dx = \sum_{\gamma\in\mathbb{Z}} \int_{S_\gamma} f(y)\omega(y)dy.$$

Second, we also often use the fact that

$$\int_{\mathbb{Q}_p^d} f(ax)dx = \frac{1}{|a|_p^d}\int_{\mathbb{Q}_p^d} f(x)dx,$$

if $a \in \mathbb{Q}_p^d \setminus \{0\}$ and $f \in L^1(\mathbb{Q}_p^d)$.

In order to prove the main theorem, we need the following lemma.

Lemma 2.2. Let $\omega \in \mathcal{W}_\alpha, \alpha > -d$ and $\gamma > 0$. Then, the funtions

$$f_{r,\gamma}(x) = \begin{cases} 0 & \text{if } |x|_p < 1 \\ |x|_p^{-\frac{d+\alpha}{r}-\frac{1}{\gamma^2}} & \text{if } |x|_p \geq 1 \end{cases}$$

belong to $L_\omega^r(\mathbb{Q}_p^d)$ and $\|f_{r,\gamma}\|_{L_\omega^r(\mathbb{Q}_p^d)} = \left(\frac{\omega(S_0)}{1-p^{-r/\gamma^2}}\right)^{1/r} > 0$

Proof. From the formula for $f_{r,\gamma}$, we see that

$$\begin{aligned}\|f_{r,\gamma}\|_{L_\omega^r(\mathbb{Q}_p^d)}^r &= \int_{\mathbb{Q}_p^d} |f_{r,\gamma}|^r \omega(x)dx \\ &= \int_{|x|p\geq 1} |x|_p^{-\left(d+\alpha+\frac{r}{\gamma^2}\right)}\omega(x)dx \\ &= \sum_{k=0}^{\infty}\int_{S_k} p^{-k\left(d+\alpha+\frac{r}{\gamma^2}\right)}\omega(x)dx \\ &= \sum_{k=0}^{\infty}\int_{S_0} p^{-k\left(d+\alpha+\frac{r}{\gamma^2}\right)}p^{k\alpha+kd}\omega(y)dy \\ &= \sum_{k=0}^{\infty} p^{-\frac{kr}{\gamma^2}}\omega(S_0) \\ &= \frac{1}{1-p^{-\frac{r}{\gamma^2}}}\omega(S_0) \\ &< \infty\end{aligned}$$

Thus $f_{r,\gamma} \in L_\omega^r(\mathbb{Q}_p^d)$ for each γ and $\|f_{r,\gamma}\|_{L_\omega^r(\mathbb{Q}_p^d)} = \left(\frac{\omega(S_0)}{1-p^{-r/\gamma^2}}\right)^{1/r} > 0.$

2.2 . Bounds of $U_{\psi,\vec{s}}^{p,2,n}$ on the product of weighted Lebesgue spaces

Let X be $L_\omega^q(\mathbb{Q}_p^d)$. Our aim is to characterize condition on functions $\psi(t)$ and $s_1(t), s_2(t)$ such that

$$\left\|U_{\psi,\vec{s}}^{p,2,n}(f_1, f_2)\right\|_{X\times X} \leq C\|f_1\|_X \cdot \|f_2\|_X$$

holds for any f_1, f_2 and the best constant C is obtained. The main result of this section is Theorem 3.2.

In this section, if not explicitly stated otherwise, $q, \alpha, q_1, q_2, \alpha_1, \alpha_2$ are real numbers, $1 \leq q < \infty, 1 \leq q_1 < \infty, 1 \leq q_2 < \infty, \alpha_1 > -d, \alpha_2 > -d$ so that

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},$$

and

$$\alpha = \frac{q\alpha_1}{q_1} + \frac{q\alpha_2}{q_2}.$$

The weights $\omega_1 \in \mathcal{W}_{\alpha_1}, \omega_2 \in \mathcal{W}_{\alpha_2}$, set

$$\omega(x) = \omega_1^{\frac{q}{q_1}}(x) \cdot \omega_2^{\frac{q}{q_2}}(x)$$

It is obvious that $\omega \in \mathcal{W}_\alpha$

Definition 3.1. We say that (ω_1, ω_2) satisfies the $\mathcal{W}_{\vec{\alpha}}$ condition if

$$\omega(S_0) \geq \omega_1(S_0)^{\frac{q}{q_1}} \omega_2(S_0)^{\frac{q}{q_2}}$$

For example, (ω_1, ω_2) where $\omega_1(x) = |x|_p^{\alpha_1}, \omega_2(x) = |x|_p^{\alpha_2}$ is satisfies the $\mathcal{W}_{\vec{\alpha}}$ condition.

Through out this paper, s_1, s_2 are measurable functions from $\left(\mathbb{Z}_p^\star\right)^n$ into $\mathbb{Q}_p$ and we denote by $\vec{s}$ the vector (s_1, s_2).

Theorem 3.2. Assume that (ω_1, ω_2) satisfies $\mathcal{W}_{\vec{\alpha}}$ condition and there exits constant $\beta > 0$ such that $|s_1(t_1, \dots, t_n)|_p \geq \min\left\{|t_1|_p^\beta, \dots, |t_n|_p^\beta\right\}$ and $|s_2(t_1, \dots, t_n)|_p \geq \min\left\{|t_1|_p^\beta, \dots, |t_n|_p^\beta\right\}$ and for almost everywhere $(t_1, \dots, t_n) \in \left(\mathbb{Z}_p^\star\right)^n$. Then there exists a constant C such that the inequality

$$\left\|U_{\psi,\vec{s}}^{p,2,n}(f_1, f_2)\right\|_{L_\omega^q(\mathbb{Q}_p^d)} \leq C\|f_1\|_{L_{\omega_1}^{q_1}(\mathbb{Q}_p^d)} \cdot \|f_2\|_{L_{\omega_2}^{q_2}(\mathbb{Q}_p^d)}$$

holds for any measurable f_1, f_2 if and only if

$$\mathcal{A} := \int_{(\mathbb{Z}_p^\star)^n} |s_1(t)|_p^{-\frac{d+\alpha_1}{q_1}} \cdot |s_2(t)|_p^{-\frac{d+\alpha_2}{q_2}} \psi(t) dt < \infty$$

Moreover, if (3.7) holds then $\mathcal{A}$ is the norm of $U_{v,\vec{s}}^{p,2,n}$ from $L_{\omega_1}^{q_1}\left(\mathbb{Q}_p^d\right) \times L_{\omega_2}^{q_2}\left(\mathbb{Q}_p^d\right)$ to $L_\omega^q\left(\mathbb{Q}_p^d\right)$.

Proof. As we note above, $\omega \in \mathcal{W}_{\vec{\alpha}}$. Firstly, suppose that $\mathcal{A}$ is finite. Let $f_1 \in L_{\omega_1}^{q_1}\left(\mathbb{Q}_p^d\right), f_2 \in L_{\omega_2}^{q_2}\left(\mathbb{Q}_p^d\right)$. Using Minkowski's inequality, Hölder's inequality and p-adic change of variable (2.2), we have

$$
\begin{aligned}
&\left\|U_{\psi,\vec{s}}^{p,2,n}(f_1,f_2)\right\|_{L_\omega^q(\mathbb{Q}_p^d)}\\
\leq\ & \iint_{\mathbb{Q}_p^d}\left(\left(\int_{(\mathbb{Z}_p^\star)^n}(|f_1(s_1(t)x)f_2(s_2(t)x)|)\psi(t)dt\right)^q\omega(x)dx\right)^{\frac{1}{q}}\\
\leq\ & \int_{(\mathbb{Z}_p^\star)^n}\left(\int_{\mathbb{Q}_p^d}|f_1(s_1(t)x)f_2(s_2(t)x)|^q\omega(x)dx\right)^{\frac{1}{q}}\psi(t)dt\\
\leq\ & \int_{(\mathbb{Z}_p^\star)^n}\prod_{k=1}^{2}\left(\int_{\mathbb{Q}_p^d}|f_k(s_k(t)x)|^{q_k}\omega_k(x)dx\right)^{\frac{1}{q_k}}\psi(t)dt\\
=\ & \mathcal{A}\left(\prod_{k=1}^{2}\|f_k\|_{L_{\omega_k}^{q_k}(\mathbb{Q}_p^d)}\right)<\infty.
\end{aligned}
$$

Thus, $U_{\psi,\vec{s}}^{p,2,n}$ is bounded from $L_{\omega_1}^{q_1}(\mathbb{Q}_p^d)\times L_{\omega_2}^{q_2}(\mathbb{Q}_p^d)$ to $L_\omega^q(\mathbb{Q}_p^d)$ and the best constant C in (3.6) satisfies

$$C\leq\mathcal{A}.$$

For the converse, assuming that $U_{\psi,\vec{s}}^{p,2,n}$ is defined as a bounded operator from $L_{\omega_1}^{q_1}(\mathbb{Q}_p^d)\times L_{\omega_2}^{q_2}(\mathbb{Q}_p^d)$ to $L_\omega^q(\mathbb{Q}_p^d)$. Let γ be an arbitrary positive number and we set

$$\gamma_1:=\sqrt{\frac{q_1}{q}}\gamma\ \text{ and }\ \gamma_2:=\sqrt{\frac{q_2}{q}}\gamma$$

and

$$f_{q_1,\gamma_1}=\begin{cases}0 & \text{if } |x|_p\leq 1\\ |x|_p^{-\frac{d+\alpha_1}{q_1}-\frac{1}{\gamma_1^2}} & \text{if } |x|_p\geq 1.\end{cases}$$

$$f_{q_2,\gamma_2}=\begin{cases}0 & \text{if } |x|_p\leq 1\\ |x|_p^{-\frac{d+\alpha_1}{q_1}-\frac{1}{\gamma_2^2}} & \text{if } |x|_p\geq 1.\end{cases}$$

From Lemma 2.2, we get that $f_{q_1,\gamma_1}\in L_{\omega_1}^{q_1}(\mathbb{Q}_p^d), f_{q_2,\gamma_2}\in L_{\omega_2}^{q_2}(\mathbb{Q}_p^d)$ and $\|f_{q_1,\gamma_1}\|_{L_{\omega_1}^{q_1}(\mathbb{Q}_p^d)}=\left(\frac{\omega_1(S_0)}{1-p^{-\frac{q_1}{\gamma_1^2}}}\right)^{\frac{1}{q_1}}>0, \|f_{q_2,\gamma_2}\|_{L_{\omega_2}^{q_2}(\mathbb{Q}_p^d)}=\left(\frac{\omega_2(S_0)}{1-p^{-\frac{q_2}{\gamma_2}}}\right)^{\frac{1}{q_2}}>0.$

We fix $x\in\mathbb{Q}_p^d$ which $|x|_p\geq 1$ and set

$$S_x=\left\{t\in(\mathbb{Z}_p^\star)^n:|s_1(t)x|_p>1\right\}\cap\left\{t\in(\mathbb{Z}_p^\star)^n:|s_2(t)x|_p>1\right\}.$$

From the assumption $|s_1(t_1,\dots,t_n)|_p \geq \min\left\{|t_1|_p^{\beta},\dots,|t_n|_p^{\beta}\right\}, |s_2(t_1,\dots,t_n)|_p \geq \min\left\{|t_1|_p^{\beta},\dots,|t_n|_p^{\beta}\right\}$ a.e $t=(t_1,\dots,t_n)\in\left(\mathbb{Z}_p^{\star}\right)^n$, there exist a subset E of $\left(\mathbb{Z}_p^{\star}\right)^n$ has measure zero and S_x is contained in

$$\left\{t\in\left(\mathbb{Z}_p^{\star}\right)^n : |t|_p \geq |x|_p^{-1/\beta}\right\}\setminus E.$$

Consequently, we have

$$\begin{aligned}
&\left\|U_{\psi,\vec{s}}^{p,2,n}\left(f_{q_1,\gamma_1},f_{q_2,\gamma_2}\right)\right\|_{L_\omega^q(\mathbb{Q}_p^d)}^q \\
&=\int_{(\mathbb{Q}_p^d)}\left|\int_{(\mathbb{Z}_p^{\star})^n}\left(f_{q_1,\gamma_1}(s_1(t)x)\cdot f_{q_2,\gamma_2}(s_2(t)x)\right)\psi(t)dt\right|^q\omega(x)dx \\
&=\int_{|x|_p\geq 1}\left(|x|_p^{-\frac{d+\alpha_1}{q_1}-\frac{1}{\gamma_1^2}}\cdot|x|_p^{-\frac{d+\alpha_2}{q_2}-\frac{1}{\gamma_2^2}}\right)^q\times \\
&\times\left|\int_{S_x}\left|s_1(t)\right|_p^{-\frac{d+\alpha_1}{q_1}-\frac{1}{\gamma_1^2}}\cdot|s_2(t)|_p^{-\frac{d+\alpha_2}{q_2}-\frac{1}{\gamma_2^2}}\psi(t)dt\right|^q\omega(x)dx \\
&\geq\int_{|x|_p\geq 1}|x|_p^{-d-\alpha-\frac{q}{\gamma^2}}\left(\int_{S_x}|s_1(t)|_p^{-\frac{d+\alpha_1}{q_1}-\frac{1}{\gamma_1^2}}\cdot|s_2(t)|_p^{-\frac{d+\alpha_2}{q_2}-\frac{1}{\gamma_2^2}}\psi(t)dt\right)^q\omega(x)dx \\
&\geq\int_{|x|_p\geq p^{\gamma}}|x|_p^{-d-\alpha-\frac{q}{\gamma^2}}\omega(x)dx\left(\int_{S_x}|s_1(t)|_p^{-\frac{d+\alpha_1}{q_1}}-\frac{1}{\gamma_1^2}\cdot|s_2(t)|_p^{-\frac{d+\alpha_2}{q_2}-\frac{1}{\gamma_2^2}}\psi(t)dt\right)^q \\
&=p^{-\frac{q}{\gamma}}\|f_{q,\gamma}\|_{L_\omega^q(\mathbb{Q}_p^d)}\left(\int_{S_x}|s_1(t)|_p^{-\frac{d+\alpha_1}{q_1}-\frac{1}{\gamma_1^2}}\cdot|s_2(t)|_p^{-\frac{d+\alpha_2}{q_2}-\frac{1}{\gamma_2^2}}\psi(t)dt\right)^q
\end{aligned}$$

Here we denote F by the set $\left\{t\in\left(\mathbb{Z}_p^{\star}\right)^n : |t|_p\geq p^{-\gamma/\beta}\right\}$. Since $|s_1(t_1,\dots,t_n)|_p \geq \min\left\{|t_1|_p^{\beta},\dots,|t_n|_p^{\beta}\right\}, |s_2(t_1,\dots,t_n)|_p \geq \min\left\{|t_1|_p^{\beta},\dots,|t_n|_p^{\beta}\right\}$ a.e $t=(t_1,\dots,t_n)\in\left(\mathbb{Z}_p^{\star}\right)^n$, imply that $S_x\supset F$.

Thus we have the following inequality

$$\begin{aligned}
\int_F |s_1(t)|_p^{-\frac{d+\alpha_1}{q_1}-\frac{1}{\gamma_1^2}}\cdot|s_2(t)|_p^{-\frac{d+\alpha_2}{q_2}-\frac{1}{\gamma_2^2}}\psi(t)dt &\leq p^{\frac{1}{\gamma}}\frac{\left\|U_{\psi,\vec{s}}^{p,2,n}\left(f_{q_1,\gamma_1},f_{q_2,\gamma_2}\right)\right\|}{\left\|f_{q_1,\gamma_1}\right\|_{L_{\omega_1}^{q_1}(\mathbb{Q}_p^d)}\cdot\left\|f_{q_2,\gamma_2}\right\|_{L_{\omega_2}^{q_2}(\mathbb{Q}_p^d)}} \\
&\leq Cp^{\frac{1}{\gamma}}.
\end{aligned}$$

Here C is the constant in (3.6). Letting γ to infinity, by Lebesgue's dominated convergence Theorem, we obtain

$$\int_{(\mathbb{Z}_p^\star)^n} |s_1(t)|_p^{-\frac{d+\alpha_1}{q_1}} \cdot |s_2(t)|_p^{-\frac{d+\alpha_2}{q_2}} \psi(t)dt \le C.$$

From (3.7) and (3.9), we obtain $\left\|U_{\psi,\vec{s}}^{p,2,n}\right\|_{L_{\omega_1}^{q_1}(\mathbb{Q}_p^d)\times L_{\omega_2}^{q_2}(\mathbb{Q}_p^d)\to L_\omega^q(\mathbb{Q}_p^d)} = \mathcal{A}.$

3. CONCLUSION

In this paper, we find out the norm of p-adic weighted bilinear Hardy-Cesàro operator on product of p-adic weighted Lebesgue spaces as following:

$$\left\|U_{\psi,\vec{s}}^{p,2,n}\right\|_{L_{\omega_1}^{q_1}(\mathbb{Q}_p^d)\times L_{\omega_2}^{q_2}(\mathbb{Q}_p^d)\to L_\omega^q(\mathbb{Q}_p^d)} = \int_{(\mathbb{Z}_p^\star)^n} |s_1(t)|_p^{-\frac{d+\alpha_1}{q_1}} \cdot |s_2(t)|_p^{-\frac{d+\alpha_2}{q_2}} \psi(t)dt < \infty.$$

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CHUẨN CỦA TOÁN TỬ SONG TUYẾN TÍNH p-ADIC HARDY-CESÀRO CÓ TRỌNG TRÊN TÍCH CÁC KHÔNG GIAN LEBESGUE

Tóm tắt: *Trong bài báo này, mục đích của chúng tôi là nghiên cứu tính bị chặn của toán tử $U_{\psi,\vec{s}}^{p,2,n}$ trên tích của các không gian p-adic Lebesgue có trọng. Chúng tôi tìm ra được điều kiện cần và đủ cho các hàm trọng để toán tử này bị chặn trên tích các không gian p-adic Lebesgue có trọng. Hơn nữa, chúng tôi cũng tìm ra chuẩn của toán tử song tuyến tính p-adic Hardy-cesàro tương ứng.*

Từ khoá: *Không gian Lebesgue, điều kiện.*