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A PROJECTION ALGORITHM FOR FINDING A COMMON SOLUTION OF MULTIVALUED VARIATIONAL INEQUALITY PROBLEMS AND FIXED POINT PROBLEMS

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Abstract:

In this paper, we introduce a new approximate projection algorithm for finding a common solution of multivalued variational inequality problems and fixed point problems in a real Hilbert space. The proposed algorithm combines the approximate projection method with the Halpern iteration technique. The strongly convergent theorem is established under mild conditions.

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THUẬT TOÁN CHIẾU TÌM NGHIỆM CHUNG CỦA CÁC BÀI TOÁN BẤT ĐẲNG THỨC BIẾN PHÂN ĐA TRỊ VÀ BÀI TOÁN ĐIỂM BẤT ĐỘNG

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Thông tin bài viết

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Từ khóa:
Bất đẳng thức biến phân đa trị, liên tục Lipschitz, tựa đơn điệu, phương pháp chiếu gần đúng, bài toán điểm bất động.

Tóm tắt:

Trong bài báo này, chúng tôi đưa ra một thuật toán chiếu gần đúng mới để tìm nghiệm chung của các bài toán bất đẳng thức biến phân đa giá trị và các bài toán tìm điểm bất định trong không gian Hilbert thực. Thuật toán của chúng tôi kết hợp phương pháp chiếu gần đúng với kỹ thuật lặp Halpern. Định lý hội tụ mạnh được thiết lập trong điều kiện nhẹ.

1 INTRODUCTION

Let $\mathcal{H}$ be real Hilbert space and C be nonempty, closed and convex subset of $\mathcal{H}$. The multivalued variational inequality problem for a operator $F : \mathcal{H} \to 2^{\mathcal{H}}$ such that $F(x)$ is nonempty closed convex for each $x \in \mathcal{H}$ (shortly, (MVI)), is stated as

$$\text{Find } (x^*, w^*) \in C \times F(x^*) \text{ s.t. } \langle w^*, x - x^* \rangle \geq 0$$

for all $x \in C$. From now on, one denotes the solution set of the above by $S(MVI)$. When $F : \mathcal{H} \to \mathcal{H}$ is a single-value mapping, it is the form of the following classical variational inequality problem (shortly, (VI)):

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0 \;\; \forall x \in C.$$

Mathematically, Problem (VI) can be considered as a generalized model of various known problems such as optimization problems, complementary problems, and fixed point problems. Many iterative methods have been proposed, among them, the projection and the extragradient algorithms are widely (see [1, 3, 5]). Note that the projection methods often require too harsh assumptions to obtain convergence theorems, such as the strong monotonicity or inverse strong monotonicity of the mapping F. To obtain the convergence results of the projection algorithms, Korpelevich [7] introduced an extragradient for Problem (MVI). The author showed that the algorithm is convergent when F is monotone and L-Lipschitz continuous. Afterward, Korpelevich's extragradient method has been extended and improved by many mathematicians in different ways. However, the extragradient algorithms often require computing two projections onto the feasible set C at each iteration. This can be computationally expensive when the set C is not so simple.

In [2], authors introduced an approximate projection algorithm, that only uses one projection, for solving multivalued variational inequalities involving pseudomonotone and Lipschitz continuous multivalued cost mappings in a real Hilbert space. This algorithm combines the approximate projection method with the Halpern iteration technique. The strongly convergent theorems are established under standard assumptions imposed on cost mappings. Motivated and inspired by the approximate projection method in [2], and using the Halpern iteration technique in [8], the purpose of this paper is to propose a new projection algorithm for finding a common element of the solution sets of Problem (MVI) and the set of fixed points of a finite system of demicontractive mappings S_j $(j \in J)$, namely:

$$\text{Find } x^* \in \cap_{j\in J} Fix(S_j) \cap S(MVI).$$

We have proved that the proposed algorithm is strongly convergent under the assumption of the pseudomonotonicity and Lipschitz continuity of cost mappings.

The remaining part of the paper is organized as follows. Section 2 shows preliminaries, some lemmas that will be used in proving the convergence of our proposed algorithm. The approximate projection algorithm and its convergence analysis are presented in Section 3.

2 PRELIMINARIES

The metric projection from $\mathcal{H}$ onto C is denoted by P_C and

$$P_C(x) = argmin\{\|x-y\| : \ y \in C\} \ x \in \mathcal{H}.$$

It is well known that the metric projection $P_C(\cdot)$ has the following basic property:

$$\langle x - P_C(x), y - P_C(x)\rangle \le 0, \ \forall x \in \mathcal{H}, \ y \in C.$$

Definition 2.1. A multi-valued mapping $F : \mathcal{H} \to 2^{\mathcal{H}}$ is called to be

(i) *pseudo-monotone*, if $\langle v, x-y\rangle \ge 0$ implies $\langle u, x-y\rangle \ge 0, \ \forall x, y \in \mathcal{H}, \forall u \in F(x), \ \forall y \in F(y)$;

(ii) L- *Lipschitz-continuous*, if $\rho(F(x), F(y)) \le L\|x-y\|, \ \forall x, y \in \mathcal{H}$, where ρ denotes the Hausdorff distance. By the definition, the Hausdorff distance of two sets A and B is defined as

$$\rho(A, B) = \max\{d(A, B), \ d(B, A)\},$$

where $d(A,B) = \sup_{a\in A} \inf_{b\in B} \|a-b\|$, $d(B,A) = \sup_{b\in A} \inf_{a\in A} \|a-b\|$.

Definition 2.2. Let $C \subset \mathcal{H}$ be a nonempty subset. An operator $S : C \to \mathcal{H}$ is called to be
(i) β-*demi-contractive* on C, if $Fix(S)$ is nonempty and there exists $\beta \in [0, 1)$ such that

$$\|Sx-p\|^2 \le \|x-p\|^2 + \beta\|x-Sx\|^2, \qquad (1)$$

for all $x \in C$ and $p \in Fix(S)$;
(ii) *demi-closed*, if for any sequence $\{x^k\} \subset C$, $x^k \rightharpoonup z \in C$, $(I-S)(x^k) \rightharpoonup 0$ implies $z \in Fix(S)$.

It is well known that if S is β-demi-contractive on C then S is demi-closed and (1) is equivalent to (see [10])

$$\langle x - Sx, x-p\rangle \ge \frac{1}{2}(1-\beta)\|x-Sx\|^2, \qquad (2)$$

for all $x \in C$ and $p \in Fix(S)$.

The following lemmas are useful in the sequel.

Lemma 2.3. *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the following condition:*

$$a_{k+1} \le (1-\alpha_k)a_k + \alpha_k\alpha_k + \gamma_k, \ \forall k \ge 1,$$

where $\{\alpha_k\} \subset [0,1]$, $\sum_{k=0}^{\infty} \alpha_k = +\infty$, $\limsup \alpha_k \le 0$, *and* $\gamma_k \ge 0$, $\sum_{n=1}^{\infty} \gamma_k < \infty$. *Then,* $\lim\limits_{n\to\infty} a_k = 0$.

Lemma 2.4. *([4], Theorem 2.1.3) Let C be a convex subset of a real Hilbert space $\mathcal{H}$ and $g : C \to \mathcal{R} \cup \{+\infty\}$ be subdifferentiable. Then, $\bar{x}$ is a solution to the following convex problem:*

$$\min\{g(x) : x \in C\}$$

if and only if $0 \in \partial g(\bar{x}) + N_C(\bar{x})$, where ∂g denotes the subdifferential of g and $N_C(\bar{x})$ is the outer normal cone of C at $\bar{x} \in C$.

Lemma 2.5. *([9], Remark 4.4) Let $\{a_k\}$ be a sequence of nonnegative real numbers. Suppose that for any integer m, there exists an integer p such that $p \ge m$ and $a_p \le a_{p+1}$. Let k_0 be an integer such that $a_{k_0} \le a_{k_0+1}$ and define, for all integer $k \ge k_0$,*

$$\tau(k) = \max\{i \in \mathcal{N} : k_0 \le i \le k, a_i \le a_{i+1}\}.$$

Then, $0 \le a_k \le a_{\tau(k)+1}$ for all $k \ge k_0$. Furthermore, the sequence $\{\tau(k)\}_{k\ge k_0}$ is nondecreasing and tends to $+\infty$ as $k \to \infty$.

3 APPROXIMATE PROJECTION ALGORITHM

Let us assume that the cost mapping $F : \mathcal{H} \to 2^{\mathcal{H}}$ and mappings S_j satisfy the following conditions:

A_1. F is pseumonotone, L-Lipschitz continuous on $\mathcal{H}$;

A_2. $S_j : \mathcal{H} \to \mathcal{H}$ is β_j-demicontractive for every $j \in J$;

A_3. $\cap_{j\in J} Fix(S_j) \cap S(MVI) \neq \emptyset$

A_4. F satisfies following property: if $x^k \rightharpoonup \bar{x}$ and $w^k \in F(x^k)$, then exists a subsequence $\{w^{k_j}\}$ of $\{w^k\}$ such that $w^{k_j} \rightharpoonup \bar{w} \in F(\bar{x})$.

Now, we describe our approximate projection algorithm.

Algorithm 3.1. *Choose starting point $x^0 \in \mathcal{H}$, $\bar{L} > L$, sequences $\{\alpha_k\}, \{\lambda_k\}$ and $\{\eta_k\}$ such that*

$$\begin{cases} \{\alpha_k\} \subset (0,1), \lim\limits_{k\to\infty} \alpha_k = 0, \sum_{k=0}^{\infty} \alpha_k = +\infty, \\ 0 < \eta_k \le \alpha_k^3, \sum_{k=0}^{\infty} \eta_k^{\frac{1}{4}} < \infty, \eta_k \le \frac{1}{\rho_k^2} \text{ if } \rho_k > 0, \\ \{\lambda_k\} \subset [a,b] \subset \left(0, \frac{1}{L}\right) \subset (0,\infty). \end{cases} \tag{3}$$

Step 1. $(k = 0,1,...)$ *Take $u^k \in F(x^k)$. Find $y^k \in C$ such that*

$$\langle y^k - x^k + \lambda_k u^k, x - y^k\rangle \ge -\eta_k \quad \forall x \in C.$$

Step 2. *Take $v^k \in B\left(u^k, \bar{L}\|x^k - y^k\|\right) \cap F(y^k)$, where $B\left(u^k, \bar{L}\|x^k - y^k\|\right) := \{x \in \mathcal{H} : \|x - u^k\| \le \bar{L}\|x^k - y^k\|\}$. Set $d^k := x^k - y^k - \lambda_k(u^k - v^k)$ and $w^k := x^k - \gamma\rho_k d(x^k, y^k)$, $\forall k \ge 0$, with $\gamma \in (0,2)$ and*

$$\rho_k = \begin{cases} \frac{\langle x^k - y^k, d(x^k,y^k)\rangle}{\|d^k\|^2}, & d^k \neq 0 \\ 0, & d^k = 0. \end{cases} \tag{4}$$

Step 3. *Compute*

$$p^k = \alpha_k x^0 + (1-\alpha_k) w^k,$$

$$q_j^k = (1-\omega)p^k + \omega S_j p^k, \ 0 < \omega < \frac{1-\beta_j}{2},$$

for all $j \in J$,

$$x^{k+1} = q_{j_0}^k, \ j_0 = argmax\{||q_j^k - p^k||, \ j \in J\}. \tag{5}$$

Step 4. *Set $k := k+1$, and go to Step 1.*

Lemma 3.1. *(see [2]) Let two sequences $\{x^k\}$ and $\{y^k\}$ be defined by the algorithm 3.1. The following inequalities hold*

$$\langle x^k - y^k, d^k\rangle \ge c_1\|x^k - y^k\|^2 \text{ and } \langle x^k - y^k, d^k\rangle \ge c_2\|d^k\|^2.$$

Lemma 3.2. *Let $x^* \in S(MVI)$. Then,*

$$\|w^k - x^*\|^2 \le \|x^k - x^*\|^2 - \frac{2-\gamma}{\gamma}\|w^k - x^k\|^2 + 2\gamma\sqrt{\eta_k}.$$

Proof. Since Step 1 and $x^* \in C$, we have $\langle y^k - x^*, x^k - y^k - \lambda_k u^k\rangle \ge -\eta_k$. Using $(x^*, w^*) \in S(MVI)$, i.e., $\langle w^*, y^k - x^*\rangle \ge 0$ and the pseudomonotone assumption of F, we get $\lambda_k\langle v^k, y^k - x^*\rangle \ge 0$. From two last inequalities, it follows

$$-\eta_k \le \langle y^k - x^*, x^k - y^k - \lambda_k u^k + \lambda_k v^k\rangle = \langle y^k - x^*, d^k\rangle.$$

Using this inequality, Condition (3) and Step 2, we have

$$\begin{aligned} &\|w^k - x^*\|^2 \\ =&\|x^k - \gamma\rho_k d^k - x^*\|^2 \\ =&\|x^k - x^*\|^2 - 2\gamma\rho_k\langle x^k - x^*, d^k\rangle + \gamma^2\rho_k^2\|d^k\|^2 \\ \le&\|x^k - x^*\|^2 - 2\gamma\rho_k\langle x^k - y^k, d^k\rangle + \gamma^2\rho_k^2\|d^k\|^2 \\ &+ 2\gamma\rho_k\eta_k \\ =&\|x^k - x^*\|^2 - 2\gamma\rho_k\langle x^k - y^k, d^k\rangle + \gamma^2\rho_k\langle x^k - y^k, d^k\rangle \\ &+ 2\gamma\rho_k\eta_k \\ =&\|x^k - x^*\|^2 - \frac{2-\gamma}{\gamma}\|w^k - x^k\|^2 + 2\gamma\rho_k\eta_k \\ \le&\|x^k - x^*\|^2 - \frac{2-\gamma}{\gamma}\|w^k - x^k\|^2 + 2\gamma\sqrt{\eta_k}. \end{aligned} \tag{6}$$

□

Lemma 3.3. *The sequences $\{p^k\}, \{x^k\}$ and $\{w^k\}$ are bounded.*

Proof. Let $x^* \in \cap_{j\in J} Fix(S_j) \cap Sol(C,F)$. Using Step 3 and the β_j demi-contractive assumption of S_j, $j = 1,2,...$, we get

$$\begin{aligned} &||x^{k+1} - x^*||^2 \\ =&||(1-\omega)p^k + \omega S_{j_0}p^k - x^*||^2 \\ =&||(p^k - x^*) + \omega(S_{j_0}p^k - p^k)||^2 \\ \le&||p^k - x^*||^2 + 2\omega\langle p^k - x^*, S_{j_0}p^k - p^k\rangle \\ &+ \omega^2||S_{j_0}p^k - p^k||^2 \\ \le&||p^k - x^*||^2 + \omega(\omega + \beta_{j_0} - 1)||S_{j_0}p^k - p^k||^2 \\ \le&||p^k - x^*||^2. \end{aligned} \tag{7}$$

From Lemma 3.2 and the last inequality, it follows that

$$||w^{k+1} - x^*|| \le ||p^k - x^*|| + 2\eta_{k+1}^{\frac{1}{4}}. \qquad (8)$$

Using Step 3, Condition (3) and (8), we have

$$\begin{aligned}
&\|p^{k+1} - x^*\| \\
=&||\alpha_{k+1}(x^0 - x^*) + (1-\alpha_{k+1})(w^{k+1} - x^*)|| \\
\le&\alpha_{k+1}||x^0 - x^*|| + (1-\alpha_{k+1})||w^{k+1} - x^*|| \\
\le&\alpha_{k+1}||x^0 - x^*|| + (1-\alpha_{k+1})(||p^k - x^*|| + 2\eta_{k+1}^{\frac{1}{4}}) \\
\le&\max\{||p^k - x^*|| + 2\eta_{k+1}^{\frac{1}{4}},\ ||x^0 - x^*||\} \\
&\dots \\
\le&\max\{||p^0 - x^*|| + 2\sum_{i=1}^{k+1}\eta_i^{\frac{1}{4}},\ ||x^0 - x^*||\} < +\infty \\
\le&\max\{||p^0 - x^*||,\ ||x^0 - x^*||\} + 2\sum_{i=1}^{\infty}\eta_i^{\frac{1}{4}} < +\infty.
\end{aligned}$$

So, the sequence $\{p^k\}$ is bounded. From (7) and (8), it follows that the sequences $\{x^k\}$ and $\{w^k\}$ are bounded. □

Lemma 3.4. *Let* $x^* \in \cap_{j\in J} Fix(S_j) \cap Sol(C, F)$. *Set* $a_k = \|x^k - x^*\|^2$, $\gamma_k = 2\gamma\sqrt{\eta_k}$ *and* $b_k = 2\langle x^0 - x^*, p^k - x^*\rangle$. *Then,*

(i) $a_{k+1} \le (1-\alpha_k)a_k + \alpha_k b_k + \gamma_k$;

(ii) $\gamma_k \ge 0,\ \sum_{n=1}^{\infty}\gamma_k < \infty$;

(iii) $\lim\limits_{k\to\infty}\frac{\gamma_k}{\alpha_k} = 0$.

Proof. Using Lemma 3.2 and Step 3, we get

$$\begin{aligned}
&||p^k - x^*||^2 \\
=&||\alpha_k(x^0 - x^*) + (1-\alpha_k)(w^k - x^*)||^2 \\
\le&(1-\alpha_k)||w^k - x^*||^2 + 2\alpha_k\langle x^0 - x^*, p^k - x^*\rangle \\
\le&(1-\alpha_k)||x^k - x^*||^2 + 2\alpha_k\langle x^0 - x^*, p^k - x^*\rangle \\
&+ 2\gamma\sqrt{\eta_k}(1-\alpha_k) \\
\le&(1-\alpha_k)||x^k - x^*||^2 + 2\alpha_k\langle x^0 - x^*, p^k - x^*\rangle \\
&+ 2\gamma\sqrt{\eta_k}. \qquad (9)
\end{aligned}$$

Using last inequality and (7), we have

$$\begin{aligned}
||x^{k+1} - x^*||^2 \le&(1-\alpha_k)||x^k - x^*|| \\
&+ 2\alpha_k\langle x^0 - x^*, p^k - x^*\rangle + 2\gamma\sqrt{\eta_k}.
\end{aligned}$$

This follows (i). Note that (ii) and (iii) are deduced from the condition (3). □

Lemma 3.5. *Suppose that* $\lim_{k\to\infty}\|x^k - y^k\| = 0$, $\lim_{k\to\infty}\|w^k - y^k\| = 0$, $\lim_{k\to\infty}\|x^{k+1} - p^k\| = 0$ *and* $x^{k_i} \rightharpoonup p$ *as* $i\to\infty$. *Then* $p \in \cap_{j\in J}Fix(S_j) \cap Sol(C, F)$.

Proof. By Step 1, we have

$$\begin{aligned}
&\langle x^{k_i} - y^{k_i}, x - y^{k_i}\rangle + \lambda_{k_i}\langle u^{k_i}, y^{k_i} - x^{k_i}\rangle \\
&\le \lambda_{k_i}\langle u^{k_i}, x - x^{k_i}\rangle + \eta_{k_i} \quad \forall x \in C.
\end{aligned}$$

For each fixed point $x \in C$, take the limit as $i \to \infty$, using $\lim_{i\to\infty}\|x^{k_i} - y^{k_i}\| = 0$ and $\lim_{i\to\infty}\eta_{k_i} = 0$, we get $\liminf_{i\to\infty}\langle u^{k_i}, x - x^{k_i}\rangle \ge 0 \quad \forall x \in C$. Let $\{\epsilon_j\}$ be a positive sequence decreasing and $\lim_{j\to\infty}\epsilon_j = 0$. Then, for each $j \in \mathcal{N}$, there exists a smallest positive integer K_j such that $\left\langle u^{K_j}, x - x^{K_j}\right\rangle + \epsilon_j \ge 0\ \ \forall x \in C$. It is easy to check that $\{K_j\}$ is increasing. Set $\nu^{K_j} := \frac{1}{\|u^{K_j}\|^2}u^{K_j}$. Then, we have $\langle u^{K_j}, \nu^{K_j}\rangle = 1$ for all $j \in \mathcal{N}$ and $\left\langle u^{K_j}, x + \epsilon_j\nu^{K_j} - x^{K_j}\right\rangle \ge 0 \quad \forall x \in C$. Combining this and the pseudomonotonicity of F, we have

$$\left\langle u, x + \epsilon_j\nu^{K_j} - x^{K_j}\right\rangle \ge 0\ \ \forall x \in C, u \in F(x+\epsilon_j\nu^{K_j}). \qquad (10)$$

Using the assumptions A_2 and $x^{K_j} \rightharpoonup p$ as $j \to \infty$, the sequence $\{u^{K_j}\}$ converges weakly to $u_p \in F(p)$. If $u_p = 0$ then (p, u_p) is a solution. So we can suppose that $u_p \ne 0$. Then, we have $0 < \|u_p\| \le \liminf_{j\to\infty}\|u^{K_j}\|$, and hence

$$\begin{aligned}
0 \le \limsup_{j\to\infty}\epsilon_j\|\nu^{K_j}\| &= \limsup_{j\to\infty}\frac{\epsilon_j}{\|u^{K_j}\|} \\
&\le \frac{\limsup_{j\to\infty}\epsilon_j}{\liminf_{j\to\infty}\|u^{K_j}\|} = 0.
\end{aligned}$$

Consequently

$$\lim_{j\to\infty}\epsilon_j\|\nu^{K_j}\| = 0. \qquad (11)$$

For each $\bar{u} \in F(x)$, set $\bar{u}^{K_j} = Pr_{F\left(x+\epsilon_j\nu^{K_j}\right)}(\bar{u})$. By the definition of the projection, we have

$$\begin{aligned}
&\|\bar{u} - \bar{u}^{K_j}\| = d\left(\bar{u}, F\left(x + \epsilon_j\nu^{K_j}\right)\right) \\
&\le \rho\left(F(x), F\left(x + \epsilon_j\nu^{K_j}\right)\right) \le L\|\epsilon_j\nu^{K_j}\|.
\end{aligned}$$

From (11) and this, it follows that

$$\lim_{j\to\infty}\|\bar{u} - \bar{u}^{K_j}\| = 0. \qquad (12)$$

Using the assumption $\lim_{k\to\infty}\|x^k - y^k\| = 0$ and $x^{K_j} \rightharpoonup p$, the sequence $\{y^{K_j}\}$ also converges weakly to p. Substituting $u := \bar{u}^{K_j} \in F\left(x + \epsilon_j\nu^{K_j}\right)$ into (10), we get

$$\left\langle \bar{u}^{K_j}, x + \epsilon_j\nu^{K_j} - x^{K_j}\right\rangle \ge 0\ \ \forall x \in C.$$

Passing the limit into the last inequality, using (12) and $\lim_{j\to\infty}\epsilon_j = 0$, we obtain $\langle\bar{u}, x - p\rangle \ge 0\ \ \forall x \in C$. For every $t \in [0, 1]$, set $x_t := tx + (1-t)p \in C$. There exists $u_t \in F(x_t)$ such that

$$0 \le \langle u_t, x_t - p\rangle = \langle u_t, tx + (1-t)p - p\rangle = t\langle u_t, x - p\rangle,$$

for all $x \in C$. Let $t \searrow 0$. By the assumption A_4, we have that $\{u_t\}$ converges weakly to $\hat{u} \in F(p)$ and hence $\langle \hat{u}, x - p\rangle \geq 0 \quad \forall x \in C$. It implies $p \in S(MVI)$. For each $j \in J$, we now show that $p \in Fix(S_j)$. Using Step 3, we have

$$\begin{aligned}||p^k - S_j p^k|| &= \frac{1}{\omega}||p^k - q_j^k|| \\ &\leq \frac{1}{\omega}||p^k - q_{j_0}^k|| = \frac{1}{\omega}||x^{k+1} - p^k||.\end{aligned}$$

From $\lim_{k\to\infty} \|x^{k+1} - p^k\| = 0$ and last inequality, it follows that $||p^k - S_j p^k|| \to 0,\ k \to \infty$. Also we know from Step 3 that

$$||p^k - w^k|| = \alpha_k||x^0 - w^k|| \leq \alpha_k M_0 \to 0,\ k \to \infty, \tag{13}$$

where $M_0 = \sup\{||x^0 - w^k|| :\ k = 0, 1, ...\}$. Using $\lim_{k\to\infty} \|x^k - y^k\| = 0$, $\lim_{k\to\infty} \|w^k - y^k\| = 0$ and $\|w^k - x^k\| \leq \|w^k - y^k\| + \|y^k - x^k\|$, we have $\lim_{k\to\infty} \|w^k - x^k\| = 0$. Combining this and (13), we obtain

$$\|p^k - x^k\| \leq \|p^k - w^k\| + \|w^k - x^k\|.$$

From this and $x^{k_i} \rightharpoonup z$, it follows that $p^{k_i} \rightharpoonup p$. Using this, $\lim_{k\to\infty} ||p^k - S_j p^k|| = 0$ and the demiclosedness of S_j, we have $p \in Fix(S_j)$. □

Theorem 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Suppose that conditions $A_1 - A_4$ are satisfied. Let $\{x^k\}$ be a sequence generated by Algorithm 3.1. Then, the sequence $\{x^k\}$ converges strongly to a solution*

$$z \in \cap_{j\in J} Fix(S_j) \cap S(MVI),$$

where $z = Pr_{\cap_{j\in J} Fix(S^j)\cap S(MVI)}(x^0)$.

Proof. Set $a_k := \|x^k - z\|$. To prove the strong convergence of the algorithm 3.1, we consider two the following cases.

Case 1. Suppose that there exists $k_0 \in \mathcal{N}$ such that $a_{k+1} \leq a_k$ for all $k \geq k_0$. There exists the limit $A = \lim_{k\to\infty} a_k \in [0, \infty)$. Using Step 3, we obtain

$$\begin{aligned}&\|x^{k+1} - z\|^2 \\ =&\|(1-\omega)p^k + \omega S_{j_0} p^k - z\|^2 \\ =&\|p^k - z\|^2 - 2\omega\langle p^k - z, p^k - S_{j_0} p^k\rangle \\ &+ \omega^2\|p^k - S_{j_0} p^k\|^2.\end{aligned} \tag{14}$$

which together with Lemma 3.2 and (2) implies that

$$\begin{aligned}&\|x^{k+1} - z\|^2 \\ \leq&\|p^k - z\|^2 - \omega(1 - \beta_{j_0} - \omega)\|p^k - S_{j_0} p^k\|^2 \\ =&||\alpha_k(x^0 - z) + (1-\alpha_k)(w^k - z)||^2 \\ &- \frac{1}{\omega}(1 - \beta_{j_0} - \omega)\|x^{k+1} - p^k\|^2 \\ \leq&(1-\alpha_k)||w^k - z||^2 + 2\alpha_k\langle x^0 - z, p^k - z\rangle \\ &- \|x^{k+1} - p^k\|^2, \\ \leq&||w^k - z||^2 + 2\alpha_k\langle x^0 - z, p^k - z\rangle - \|x^{k+1} - p^k\|^2 \\ \leq&\|x^k - z\|^2 - \frac{2-\gamma}{\gamma}\|w^k - z\|^2 + 2\alpha_k\langle x^0 - z, p^k - z\rangle \\ &- \|x^{k+1} - p^k\|^2 \\ \leq&\|x^k - z\|^2 - \frac{2-\gamma}{\gamma}\|w^k - z\|^2 + \alpha_k M_1 \\ &- \|x^{k+1} - p^k\|^2,\end{aligned} \tag{15}$$

where $M_1 := \sup\{2\langle x^0 - z, p^k - z\rangle : k = 0, 1, ...\} < \infty$. It follows that

$$\begin{aligned}&a_{k+1} - a_k + \frac{2-\gamma}{\gamma}\|w^k - x^k\|^2 + \|x^{k+1} - p^k\|^2 \\ &\leq \alpha_k M_1 + 2\gamma\sqrt{\eta_k} \quad \forall k \geq 0.\end{aligned} \tag{16}$$

Passing the limit as $k \to \infty$ and using the assumptions $\lim_{k\to\infty} \alpha_k = 0, \lim_{k\to\infty} \eta_k = 0, \gamma \in (0, 2)$, we have $\lim_{k\to\infty} \|w^k - x^k\| = 0,\ \lim_{k\to\infty} \|x^{k+1} - p^k\| = 0$. By Lemma 3.1 and Step 2, we have $\rho_k \geq c_2$ and

$$\begin{aligned}\|x^k - y^k\|^2 \leq&\frac{1}{c_1}\langle x^k - y^k, d^k\rangle \\ =&\frac{1}{c_1\rho_k\gamma^2}\|w^k - x^k\|^2 \\ \leq&\frac{1}{c_1 c_2\gamma^2}\|w^k - x^k\|^2\end{aligned}$$

Since $\lim_{k\to\infty} \|w^k - x^k\| = 0$ we get $\lim_{k\to\infty} \|x^k - y^k\| = 0$. It follows that

$$\|w^k - y^k\| \leq \|w^k - x^k\| + \|x^k - y^k\| \to 0, \text{ as } k \to \infty.$$

Using Step 3, we have $\|p^k - w^k\| = \alpha_k\|x^0 - w^k\| \leq \alpha_k M_0 \to 0$, as $k \to \infty$, where $M_0 = \sup\{\|x^0 - w^k\| :\ k = 0, 1, ...\}0 < +\infty$. Therefore,

$$\|x^{k+1} - x^k\| \leq \|x^{k+1} - p^k\| + \|p^k - w^k\| + \|w^k - x^k\| \to 0$$

as $k \to \infty$. From this and $\|x^k - p^k\| \leq \|x^{k+1} - x^k\| + \|x^{k+1} - p^k\|$, it follows that $\lim_{k\to\infty} \|x^k - p^k\| = 0$. Since sequence $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_i}\}$ such that $x^{k_i} \rightharpoonup p \in \mathcal{H}$ and $\limsup_{k\to\infty}\langle x^0 - z, x^k - z\rangle = \lim_{i\to\infty}\langle x^0 - z, x^{k_i} - z\rangle$. Using $\lim_{k\to\infty} \|x^k - y^k\| = 0$, $\|w^k - y^k\| \to 0$, $\|x^{k+1} - p^k\| \to 0$ and Lemma 3.5, we have

$p \in \cap_{j\in J} Fix(S_j) \cap Sol(C,F)$. From $\lim_{i\to\infty} \|x^{k_i} - p^{k_i}\| = 0$ and $x^{k_i} \rightharpoonup p$, it follows that $p^{k_i} \rightharpoonup p$. Therefore, we get $\limsup\limits_{k\to\infty} b_k = 2\lim\limits_{i\to\infty} \langle x^0 - z, p^{k_i} - z\rangle = 2\langle x^0 - z, p - z\rangle \le 0$. Using this, Lemma 2.3 and Lemma 3.4, we obtain $\lim\limits_{k\to\infty} \|x^k - z\| = 0$.

Case 2. Assume that there not exist $k_0 \in \mathbb{N}$ such that $\{a_k\}_{k=k_0}^{\infty}$ is monotonically decreasing. So, there exists an integer k_0 such that $a_{k_0} \le a_{k_0+1}$. By Lemma 2.5, Maingé introduced a subsequence $\{a_{\tau(k)}\}$ of $\{a_k\}$ which is defined as

$$\tau(k) = \max\{i \in \mathcal{N} : k_0 \le i \le k, a_i \le a_{i+1}\}.$$

Then, he showed that $\tau(k) \nearrow +\infty, 0 \le a_k \le a_{\tau(k)+1}, a_{\tau(k)} \le a_{\tau(k)+1} \quad \forall k \ge k_0$. Using $a_{\tau(k)} \le a_{\tau(k)+1},\ \forall k \ge k_0$ and (16), we get

$$\|w^{\tau(k)} - x^{\tau(k)}\| \to 0, \|x^{\tau(k)+1} - p^{\tau(k)}\| \to 0,\ k \to \infty.$$

By a similar way as in case 1, we can show that

$$\begin{aligned}&\lim_{k\to\infty} \|x^{\tau(k)} - p^{\tau(k)}\| = \lim_{k\to\infty} \|x^{\tau(k)} - y^{\tau(k)}\| \\ &= \lim_{k\to\infty} \|w^{\tau(k)} - y^{\tau(k)}\| = 0. \qquad (17)\end{aligned}$$

Since $\{x^{\tau(k)}\}$ is bounded, there exists a subsequence of $\{x^{\tau(k)}\}$, still denoted by $\{x^{\tau(k)}\}$, which converges weakly to $p \in \mathcal{H}$. By Lemma 3.5, we get $p \in \cap_{j\in J} Fix(S_j) \cap Sol(C,F)$. Again, by a similar way as in case 1, we can prove that $\limsup\limits_{k\to\infty} b_{\tau(k)} \le 0$. Using Lemma 3.4 (i) and $a_{\tau(k)} \le a_{\tau(k)+1},\ \forall k \ge k_0$, we have

$$\begin{aligned}&\alpha_{\tau(k)} a_{\tau(k)} \le a_{\tau(k)} - a_{\tau(k)+1} + \alpha_{\tau(k)} b_{\tau(k)} + \gamma_{\tau(k)} \\ &\le \alpha_{\tau(k)} b_{\tau(k)} + \gamma_{\tau(k)}.\end{aligned}$$

Since $\delta_{\tau(k)} > 0$, we get $a_{\tau(k)} \le b_{\tau(k)} + \frac{\gamma_{\tau(k)}}{\alpha_{\tau(k)}}$. From Lemma 3.4 (iii) and last inequality, it follows that $\limsup\limits_{k\to\infty} a_{\tau(k)} \le \limsup\limits_{k\to\infty} b_{\tau(k)} \le 0$. Hence, $\lim_{k\to\infty} a_{\tau(k)} = 0$. It follows that

$$\begin{aligned}a_{\tau(k)+1} &= \|x^{\tau(k)+1} - z\|^2 \\ &\le (\|x^{\tau(k)+1} - x^{\tau(k)}\| + \|x^{\tau(k)} - z\|)^2 \\ &\to 0,\ k \to \infty.\end{aligned}$$

Using $0 \le a_k \le a_{\tau(k)+1}$ for all $k \ge k_0$, we get $\lim\limits_{n\to\infty} a_k = 0$. Hence, $x^k \to z$ as $k \to \infty$. □

4 CONCLUSIONS

We propose a new projection algorithm for finding a common element of the solution sets of Problem (MVI) and the set of fixed points of a finite system of mappings. Our algorithm only uses one projection on C at each iteration. We show that the proposed algorithm is strongly convergent when F is pseudomonotone, Lipschitz and S_j is demicontrac- for all $j \in J$.

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