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IRREDUCIBLE DECOMPOSITION OF SQUARE OF EDGE IDEALS

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ABSTRACT

Let $R = K[x_1,...,x_d]$ be the polynomial ring in d variables over K, $G = (V(G),E(G))$ a graph associated with variables $\{x_1,...,x_d\}$ and I_G an edge ideal. In this paper, we describe the structure of irreducible decompositions of square of edge ideals I_G^2 of the polynomial ring via corner elements and coclique sets.

Key words: *Commutative Algebra; Monomial ideals; Edge ideals; Irreducible decomposition; Corrner elements; Coclique sets.*

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PHÂN TÍCH BẤT KHẢ QUY CỦA BÌNH PHƯƠNG IĐÊAN CẠNH

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TÓM TẮT

Cho $R = K[x_1,...,x_d]$ là vành đa thức d biến trên trường K, $G = (V(G),E(G))$ là đồ thị liên kết với các biến $\{x_1,...,x_d\}$ và I_G là iđêan cạnh. Trong bài báo này, chúng tôi mô tả cấu trúc của phân tích bất khả quy của bình phương của iđêan cạnh I_G^2 của vành đa thức thông qua các phần tử góc và các tập coclique.

Key words: *Đại số giao hoán; Iđêan đơn thức; Iđêan cạnh; Phân tích bất khả quy; Phần tử góc; Tập Coclique.*

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1. Introduction

Let K be a field, $R = K[x_1,...,x_d]$ the polynomial ring in d variables over K. We say that an ideal $I \subset R$ is irreducible if I cannot be written as the intersection of two larger ideals of R. When I is a monomial ideal, the set $\operatorname{Irr}(I)$ of irreducible monomial ideals appearing in such expression depends only on I. It is well known that through the structure of irreducible decompositions of I^k, we can study the asymptotic behavior of the associated primes, the depth, or the socle of I^k for $k \geqslant 2$. This problem has been studied by many authors (see [1] [2], [3], [4], [5], [6],...) Note that the structure of irreducible decompositions of I^k, for small values of k, can also be very complicated even for edge ideals. In this paper, we are interested in studying the structure of irreducible decompositions of square of edge ideals I_G^k of the polynomial ring in the case $k = 2$ via corner elements and coclique sets.

In the section 2, we will recall some results about irreducible decompositions, corner elements and coclique sets. In the section 3, we prove the main resut of the paper which describles irreducible component of powers of edge ideals I_G^2 (see Theorem 3.1) and give an example (see Example 3.2).

2. Preliminaries

In this section, we recall some terminologies that will be used in the rest of the paper. Let $R = K[x_1,\ldots,x_d]$ be a polynomial ring with d variables over the field K and $[[R]]$ the set of all monomials of R. For a non-zero vector $\mathbf{a} = (a_1,\ldots,a_d) \in \mathbb{N}^d$, we set $\mathbf{a}+\mathbf{1} = (a_1+1,\ldots,a_d+1) \in \mathbb{N}^d$, $\mathfrak{m}^{\mathbf{a}} := (x_i^{a_i} \mid i = 1,\ldots,d,\ a_i > 0)$, $\mathbf{x}^{\mathbf{a}} = x_1^{a_1}\ldots x_d^{a_d}$ and $\operatorname{Supp}(\mathbf{a}) = \operatorname{Supp}(\mathbf{x}^{\mathbf{a}}) := \{x_i \in V(G) \mid a_i \neq 0\}$.

Definition 2.1. A non-zero monomial ideal I of R is called *irreducible*, if I is of the form $\mathfrak{m}^{\mathbf{b}}$ for some non-zero vector $\mathbf{b} \in \mathbb{N}^d$. An ideal I is called $\mathfrak{m}$-*irreducible monomial ideal* if I is an irreducible ideal and $\sqrt{I} = \mathfrak{m}$. An *irreducible decomposition* of a monomial ideal I is an expression of the form $I = \mathfrak{m}^{\mathbf{b_1}} \cap \ldots \cap \mathfrak{m}^{\mathbf{b_r}}$, for some non-zero vectors $\mathbf{b_1},\ldots,\mathbf{b_r} \in \mathbb{N}^n$ and it is *irredundant*, if none of the ideals $\mathfrak{m}^{\mathbf{b_1}},...,\mathfrak{m}^{\mathbf{b_r}}$ can be dropped from the right hand side.

It is well known that if I is a monomial ideal then I has a unique irredundant irreducible decomposition $I = \cap_{i=1}^{r}\mathfrak{m}^{\mathbf{b_i}}$, the set $\{\mathfrak{m}^{\mathbf{b_1}},\ldots,\mathfrak{m}^{\mathbf{b_r}}\}$ is denoted by $\operatorname{Irr}(I)$. We also denote by $\operatorname{Irr}_{\mathfrak{m}}(I)$ the set of $\mathfrak{m}$-irreducible monomial ideals which appear in the irredundant irreducible decomposition of I.

Let $J \subset R$ be a monomial ideal and $\mu(J)$ the number of minimal generators of J.

Definition 2.2. A monomial $z \in [[R]]$ is a J-*corner element* if $z \notin J$ but $x_1 z,\ldots,x_d z \in J$. The set of corner elements of J in $[[R]]$ is denoted by $C_R(J)$.

Note that if $\operatorname{rad}(J) = \mathfrak{m}$, then it is well known that $t(R/J) = \operatorname{card}(C_R(J))$ is the type of the ring R/J. Now we need some results from [7].

Theorem 2.3. *Let $J \subset R$ be a monomial ideal.*

(i) Assume that $\operatorname{rad}(J) = \mathfrak{m}$. *Let*

$$C_R(J) = \{\mathbf{x}^{\mathbf{b_j}} \mid \mathbf{b_j} \in \mathbb{N}^d, j = 1,\ldots,t(R/J)\}$$

be the set of corner elements of J. Then $J = \cap_{j=1}^{t(R/J)}\mathfrak{m}^{\mathbf{b_j}+\mathbf{1}}$ is the unique irredundant irreducible decomposition of J.

(ii) Assume that $\operatorname{rad}(J) \neq \mathfrak{m}$ *and*

$$J = (\mathbf{x}^{\mathbf{b_j}} \mid \mathbf{b_j} \in \mathbb{N}^d, j = 1,\ldots,\mu(J))R.$$

Let m be an integer bigger or equal than any of the coordinates of the vectors $\mathbf{b_j}$. Set $J' := J + (x_1^{m+1},\ldots,x_d^{m+1})R$ and $C_R(J') = \{\mathbf{x}^{\mathbf{c_j}} \mid \mathbf{c_j} \in \mathbb{N}^d, j = 1,\ldots,t(R/J')\}$ be the set of corner elements of J'. Then $J = \cap_{j=1}^{t(R/J')}\widetilde{\mathfrak{m}^{\mathbf{c_j}+\mathbf{1}}}$

is the unique irredundant irreducible decomposition of J, where $\widetilde{\mathfrak{m}^{\mathbf{c_j}+\mathbf{1}}}$ is obtained from $\mathfrak{m}^{\mathbf{c_j}+\mathbf{1}}$ by deleting all monomials of the type $x_1^{m+1}, \ldots, x_d^{m+1}$ from its generators.

From now on, let $G = (V(G), E(G))$ be a graph with the vertex set $V(G) = \{x_1, \ldots, x_d\}$. Recall that the *edge ideal* I_G associated to G is the ideal generated by the edges of G. Note that the edge ideal I_G is a square-free monomial ideal. For each $s \leqslant d \in \mathbb{N}$, we set $S = \{x_1, \ldots, x_s\} \subset V(G)$ and $Z = V(G) \setminus S = \{z_1, \ldots, z_t\}$.

Corollary 2.4. *Let $k, m \in \mathbb{N}$ and $m \geq k$. Then the ideal $(x_1^{a_1+1}, \ldots, x_s^{a_s+1})R$ belongs to $\mathrm{Irr}(I_G^k)R$ if and only if*

$$(x_1^{a_1+1}, \ldots, x_s^{a_s+1}, z_1^{m+1}, \ldots, z_t^{m+1})$$

belongs to $\mathrm{Irr}(I_G^k + \mathfrak{m}^{\mathbf{b}})$, where $\mathbf{b} = (m+1, m+1, \ldots, m+1) \in \mathbb{N}^d$.

Note that in terms of corner elements, it is equivalent to say that the monomial $x_1^{a_1} \ldots x_s^{a_s} z_1^m \ldots z_t^m$ is a corner element of $I_G^k + \mathfrak{m}^{\mathbf{b}}$. That is

(1) $x_1^{a_1} \ldots x_s^{a_s} z_1^m \ldots z_t^m \notin I_G^k + \mathfrak{m}^{\mathbf{b}}$ but
(2) $u x_1^{a_1} \ldots x_s^{a_s} z_1^m \ldots z_t^m \in I_G^k + \mathfrak{m}^{\mathbf{b}}$ for every $u \in V(G)$.

It is clear that the second condition is immediate for $u \in Z$. The first condition implies that for any $z_i \neq z_j \in Z$, we have $z_i z_j \notin I_G$.

Definition 2.5. [8] A set $C \subset V(G)$ is a *cover* of G if for any edge $xy \in E(G)$ we have either $x \in C$ or $y \in C$. A set $S \subset V(G)$ is a *clique* of G if the induced subgraph $G[S]$ is a complete graph and it is a *coclique* of G if the induced subgraph $G[S]$ has no edges. A coclique set of G is also called *independent set*. The family of cocliques sets of G is a simplicial complex called *independent complex* of G and denoted by $\Delta(G)$.

For a set $S \subset V(G)$ we denote by $N(S)$ the set of vertices adjacent to some element in S and $\Delta_S(G)$ the family of cocliques sets of G such that $N(S) \cap Z = \emptyset$. Note that S may be not a subset of $N(S)$ and $\Delta_S(G)$ is a simplicial complex.

Remark 2.6. (i) A set $C \subset V(G)$ is a cover of G if and only if $V(G) \setminus C$ is coclique and C is a minimal cover of G if and only if $V(G) \setminus C$ is a maximal coclique.

(ii) A set $Z \subset V(G)$ is coclique if and only if $N(Z) \cap Z = \emptyset$ and Z is maximal coclique if and only if $V(G) = N(Z) \cup Z$.

Example 2.7. (i) The set Z in Corollary 2.4 is a coclique. Indeed, if there is indices $i \neq j$ such $z_i z_j$ is an edge in G then we would have $x_1^{a_1} \ldots x_s^{a_s} z_1^k \ldots z_t^k \in I_G^k + \mathfrak{m}^{\mathbf{b}}$, which is a contradiction.

(ii) As an application of the above result, let us compute the irreducible decomposition of I_G. Since it is a square free ideal, any ideal in $\mathrm{Irr}(I_G)$ is of the type $\mathfrak{m}^{\mathbf{a}}$ for some nonzero vector $\mathbf{a} \in \mathbb{N}^d$ such that $0 \leq a_i \leq 1$ for every $i = 1, \ldots, d$. Let $S = \mathrm{Supp}(\mathbf{a}), Z = V(G) \setminus S = \{z_1, \ldots, z_t\}$. Then $z_1 \ldots z_t$ is a corner element of $I_G + \mathfrak{m}^{(2,2,\ldots,2)}$, which implies that Z is a coclique. Moreover, it is a maximal coclique set in $V(G)$, since for every $u \in S$ we have $u z_1 \ldots z_t \in I_G + \mathfrak{m}^{(2,2,\ldots,2)}$, which implies that there exists some i such that $u z_i$ is an edge in G.

This proves that the irreducible (prime) ideals in $\mathrm{Irr}(I_G)$ are of the type $\mathfrak{m}^{\mathbf{a}}$ for some nonzero vector $\mathbf{a} \in \mathbb{N}^d$ with $0 \leq a_i \leq 1$ such that $V(G) \setminus \mathrm{Supp}(\mathbf{a})$ is a maximal coclique in $V(G)$. This also shows that I_G is the Stanley-Reisner ideal associated to $\Delta(G)$. Note that the set $\mathrm{Irr}(I_G)$ is also the set of minimal associated primes of I_G^k, for any $k \geq 1$.

3. Irreducible components of I_G^2

We start by recalling some definitions in [9]. A *matching* M of a graph G is a subset of E such that any two edges of M have no vertices in common. A *maximum matching* of

G is a matching that contains the largest possible number of edges. The *matching number* of a graph G, denoted by $\nu(G)$, is the number of edges in a maximum matching of G.

It is well known that if M, N are monomials without common variables and L is a list of monomials then $(MN, L) = (M, L) \cap (N, L)$. As a consequence of this fact, every irreducible component J of I_G^2 can be written $J = (y_1, \ldots, y_k, x_1^2, \ldots, x_l^2)R$, for some vertices $y_1, \ldots, y_k, x_1, \ldots, x_l$ in $V(G)$. Now put the sets $S := \{x_1, \ldots, x_l\}$, $Z = V(G) \setminus \{y_1, \ldots, y_k, x_1, \ldots, x_l\} := \{z_1, \ldots, z_m\}$.

Theorem 3.1. *Let J be an irreducible component of I_G^2 and the sets S, Z as above. Then we have either*

(i) $N(S) \cap Z = \emptyset$. In this case $\text{card}(S) = 3$, *$G[S]$ is a triangle and Z is a maximal coclique subset of $V(G) \setminus N(S)$.*

(ii) $N(S) \cap Z \neq \emptyset$. In this case $\text{card}(S) = 1$ *and Z is a maximal coclique subset of $V(G)$.*

Proof. Let $J = (y_1, \ldots, y_k, x_1^2, \ldots, x_l^2)R$ be an irreducible component of I_G^2. Then we have by Corollary 2.4 that J is an irreducible component of I_G^2 if and only if $J + (z_1^3, \ldots, z_m^3)R$ is an irreducible component of $I_G^2 + (y_1^3, \ldots, y_k^3, x_1^3, \ldots, x_l^3, z_1^3, \ldots, z_m^3)$. Therefore by term of corner elements we have $x_1 \ldots x_l z_1^2 \ldots z_m^2$ is a corner element of $I_G^2 + (y_1^3, \ldots, y_k^3, x_1^3, \ldots, x_l^3, z_1^3, \ldots, z_m^3)R$, i.e. $x_1 \ldots x_l z_1^2 \ldots z_m^2 \notin I_G^2 + (y_1^3, \ldots, y_k^3, x_1^3, \ldots, x_l^3, z_1^3, \ldots, z_m^3)R(1)$ and $u x_1 \ldots x_l z_1^2 \ldots z_m^2 \in I_G^2 + (y_1^3, \ldots, y_k^3, x_1^3, \ldots, x_l^3, z_1^3, \ldots, z_m^3)R(2)$ for every vertex u. We have two following assertions:

(a) If $m \geq 2$ then for every $1 \leq i < j \leq m$ we have $z_i z_j \notin I_G$.

(b) For every $u \notin Z$, the condition (2) implies that $u x_1 \ldots x_l z_1^2 \ldots z_m^2 \in I_G^2$. It follows that $l \geq 1$ and $x_1 \ldots x_l z_1^2 \ldots z_m^2 \in I_G$. In terms of matching number that means $\nu(S \cup Z) = 1$ and $\nu(S) \leq 1$. Now we prove the theorem.

(i) If $N(S) \cap Z = \emptyset$ then since $x_1 \ldots x_l z_1^2 \ldots z_m^2 \in I_G$ and the assertion (a), we have $x_1 \ldots x_l \in I_G$, i.e. $\nu(S) = 1$.

For $u = x_1$, we have $x_1 x_1 \ldots x_l z_1^2 \ldots z_m^2 \in I_G^2$. But since $N(S) \cap Z = \emptyset$, we have $x_1 x_1 \ldots x_l \in I_G^2$. Then there exist two edges $x_{i_1} x_{i_2}, x_{i_3} x_{i_4} \in I_G$ and they must have a common vertex, otherwise $x_1 \ldots x_l \in I_G^2$, a contradiction. Hence there exists i_1, i_2 such that $x_1 x_{i_1}, x_1 x_{i_2} \in I_G$.

Suppose that $l \geq 4$. Let x_{i_3} distinct from x_1, x_{i_1}, x_{i_2}. By using the same argument as the above, then there exists i_4, i_5 such that $x_{i_3} x_{i_4}, x_{i_3} x_{i_5} \in I_G$. We have either $x_{i_4} \neq x_1$ or $x_{i_5} \neq x_1$. Suppose $x_{i_4} \neq x_1$ and if $x_{i_4} = x_{i_1}$ then $x_1 x_{i_2}, x_{i_1} x_{i_3}$ implies $\nu(S) > 1$, a contradiction to (b), if $x_{i_4} = x_{i_2}$ then $x_1 x_{i_2}, x_{i_2} x_{i_3}$ also implies $\nu(S) > 1$, a contradiction to (b). By similar argument for the case $x_{i_5} \neq x_1$ and $x_{i_5} = x_{i_1}$ or $x_{i_5} = x_{i_2}$. So $l = 3$.

Moreover, since $x_{i_1} x_1 x_{i_1} x_{i_2} z_1^2 \ldots z_m^2 \in I_G^2$, it implies $x_{i_1} x_1 x_{i_1} x_{i_2} \in I_G^2$, and consequently $x_{i_1} x_{i_2} \in I_G$. Hence S is a triangle.

Finally, let u be a vertex such that $Z \cup \{u\}$ is coclique then $u x_1 \ldots x_l z_1^2 \ldots z_m^2 \in I_G^2$, implies $u \in N(S)$, this proves the maximality of Z inside $V(G) \setminus N(S)$.

(ii) Assume that $N(S) \cap Z \neq \emptyset$. Let $z_1 \in N(S) \cap Z$ and suppose that $x_1 z_1 \in I_G$. Then we have the following claims:

(1) $x_2 \ldots x_l \notin I_G$.

(2) $N(S) \cap Z \neq x_1$. Indeed, if there exists $i \neq 1$ such that $x_1 z_1, x_i z_j$ are two edges then $x_1 \ldots x_l z_1^2 \ldots z_m^2 \in I_G^2$, a contradiction.

(3) S has only one element. Indeed, if there exists $u \in S$ such that $u \neq x_1$ then by (2) we have $u \notin N(Z)$. Since $u x_1 \ldots x_l z_1^2 \ldots z_m^2 \in I_G^2$, there exists $v \in S$ such that $uv \in I_G$ and $x_1 \ldots \widehat{v} \ldots x_l z_1^2 \ldots z_m^2 \in I_G$. If $v \neq x_1$ then $u v x_1 z_1 \in I_G^2$, which implies that $x_1 \ldots x_l z_1^2 \ldots z_m^2 \in I_G^2$, a contradiction. If

$v = x_1$ then we have $x_2 \dots x_l z_1^2 \dots z_m^2 \in I_G$, a contradiction to (1). Thus $\text{card}(S) = 1$.

(4) Let $u \notin Z$, we have $ux_1 z_1^2 \dots z_m^2 \in I_G^2 + (y_1^3, \dots, y_k^3, x_1^3, z_1^3, \dots, z_m^3)$. Clearly, u must belong to $N(Z)$, otherwise $ux_1 z_1^2 \dots z_m^2 \notin I_G^2 + (y_1^3, \dots, y_k^3, x_1^3, z_1^3, \dots, z_m^3)$, a contradiction. Hence Z is maximal coclique subset of $V(G)$.

Example 3.2. In the figure 1 we have a graph G with $\nu(G) = 4$. We have twenty two maximal cocliques sets

$\{a, d, h, j, k\}, \{a, d, g, i\}, \{b, d, h, j, k\}$,
$\{b, d, g, i\}, \{c, d, h, j, k\}, \{c, d, i\}$,
$\{a, e, h, j, k\}, \{a, e, g, i\}, \{b, e, h, j, k\}$,
$\{b, e, g, i\}, \{c, e, h, j, k\}, \{c, e, i\}, \{a, f, h, j, k\}$,
$\{a, f, i\}, \{b, f, h, j, k\}, \{b, f, i\}, \{c, f, h, j, k\}$,
$\{c, f, i\}, \{a, d, g, j, k\}, \{b, d, g, j, k\}$,
$\{a, e, g, j, k\}, \{b, e, g, j, k\}$.

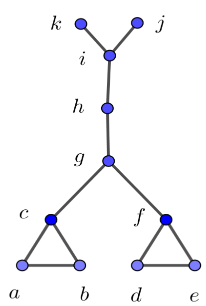

Figure 1

Hence I_G has 22 irreducible components. In this example, we have two triangles $F_1 = \{a, b, c\}, F_2 = \{d, e, f\}$. Consider for example the set F_1, there are exactly six coclique sets $Z \subset V(G) \setminus N(F_1)$ that are maximal subset of $V(G) \setminus N(F_1)$. Namely, $Z_1 = \{d, h, j, k\}, Z_2 = \{e, h, j, k\}, Z_3 = \{f, h, j, k\}, Z_4 = \{d, i\}, Z_5 = \{e, i\}, Z_6 = \{f, i\}$. This shows that

$(a^2, b^2, c^2, e, f, g, i), (a^2, b^2, c^2, d, f, g, i)$,
$(a^2, b^2, c^2, d, e, g, i), (a^2, b^2, c^2, e, f, g, h, j, k)$,
$(a^2, b^2, c^2, d, f, g, h, j, k), (a^2, b^2, c^2, d, e, g, h, j, k)$

are embedded irreducible components of I_G^2. Similarly, for F_2 there are also exactly six coclique sets. As a consequence there are exactly 12 embedded irreducible components of I_G^2. We can describe them completely.

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