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PERIODIC SOLUTIONS TO A CLASS OF DIFFERENTIAL VARIATIONAL INEQUALITIES IN BANACH SPACES

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Abstract. In this work, we consider a model formulated by a dynamical system and an elliptic variational inequality. We prove the solvability of initial value and periodic problems. Finally, an illustrative example is given to show the applicability of our results.
Keywords: Elliptic variational inequalities, periodic solution, fixed point theorems.

1. Introduction

Let $(X, \|\cdot\|_X)$ be a Banach space and $(Y, \|\cdot\|_Y)$ be a reflexive Banach space with the dual Y^*. We consider the following problem:

$$x'(t) = Ax(t) + F(t, x(t), y(t)),\ t > 0, \tag{1.1}$$
$$By(t) + \partial\phi(y(t)) \ni h(t, x(t), y(t)),\ t > 0, \tag{1.2}$$

where $(x(\cdot), y(\cdot))$ takes values in $X \times Y$; $\phi : Y \to (-\infty, \infty]$ is a proper, convex and lower semicontinuous function with the subdifferential $\partial\phi \subset Y \times Y^*$. F is a continuous function defined on $\mathbb{R}^+ \times X \times Y$. In our system, A is a closed linear operator which generates a C_0-semigroup in X; $B : Y \to Y^*$ and $h : \mathbb{R}^+ \times X \times Y \to Y^*$ are given maps which will be specified in the next section.

We study the existence of a periodic solution for this problem, that is, we find a solution of (1.1)-(1.2) with periodic condition

$$x(t) = x(t + T), \quad \text{for given } T > 0,\ \ \forall t \geq 0. \tag{1.3}$$

When F and h are autonomous maps, the system (1.1)-(1.2) was investigated in [1]. In this work, the existence of solutions and the existence of a global attractor for m-semiflow generated by solution set were proved.

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In the case $\phi = I_K$, the indicator function of K with K being a closed convex set in Y, namely,

$$I_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{otherwise}, \end{cases}$$

the problem (1.1)-(1.2) is written as follows

$$\begin{aligned} & x'(t) = Ax(t) + F(t, x(t), y(t)),\ t > 0, \\ & y(t) \in K, \forall t \geq 0, \\ & \langle By(t), z - y(t)\rangle \geq \langle h(t, x(t), y(t)), z - y(t)\rangle, \forall z \in K, t > 0. \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between Y^* and Y.

In the case $X = \mathbb{R}^n, Y = \mathbb{R}^m$ and F is single-valued, this model is a differential variational inequality (DVI), which was systematically studied by Pang and Stewart [2]. It should be mentioned that DVIs in finite dimensional spaces have been a subject of many studies in literature because they can be used to represent various models in mechanical impact problems, electrical circuits with ideal diodes, Coulomb friction problems for contacting bodies, economical dynamics, and related problems such as dynamic traffic networks. We refer the reader to [2-5] for some recent results on solvability, stability, and bifurcation to finite dimensional DVIs.

2. Main results

In this section, we consider the system (1.1)-(1.2) with initial and periodic conditions. By some suitable hypotheses imposed on given functions, we will obtain the results concerning the solvability of initial value problem and periodic problem.

2.1. The existence of solution with initial condition

We consider differential variational inequality (1.1)-(1.2) with initial datum

$$x(0) = x_0. \tag{2.1}$$

To get the existence result, we need the following assumptions.

(A) *A is a closed linear operator generating a C_0-semigroup $(S(t))_{t\geq 0}$ in X.*

(B) *B is a linear continuous operator from Y to Y^* defined by*

$$\langle u, Bv\rangle = b(u, v), \forall u, v \in Y,$$

where $b : Y \times Y \to \mathbb{R}$ is a bilinear continuous function on $Y \times Y$ such that

$$b(u, u) \geq \eta_B \|u\|_Y^2.$$

(F) $F : \mathbb{R}^+ \times X \times Y \to X$ *satisfies*

$$\|F(t,x,y) - F(t,x',y')\|_X \leq a(t)\|x - x'\|_X + b(t)\|y - y'\|_Y,$$

where $a, b \in L^1(\mathbb{R}^+; \mathbb{R}^+)$.

(H) $h : \mathbb{R}^+ \times X \times Y \to Y^*$ *is a Lipschitz continuous map. In particular, there exist two positive constants* η_{1h}, η_{2h} *and a continuous positive function* $\eta_h(\cdot,\cdot)$ *and* $\eta_h(t,t) = 0,\ \forall t \geq 0$ *such that:*

$$\|h(t,x_1,u_1) - h(t_1,x_2,u_2)\|_* \leq \eta_h(t,t_1) + \eta_{1h}\|x_1 - x_2\|_X + \eta_{2h}\|u_1 - u_2\|_Y,$$

for all $t \in \mathbb{R}^+, x_1, x_2 \in X; u_1, u_2 \in Y$, *where* $\|\cdot\|_*$ *is the norm in dual space* Y^*.

Letting $T > 0$, we mention here the definition of solution of the problem (1.1)-(1.2)-(2.1).

Definition 2.1. *A pair of continuous functions* (x, y) *is said to be a mild solution of* (1.1)-(1.2)-(2.1) *on* $[0, T]$ *if*

$$x(t) = S(t)x_0 + \int_0^t S(t-s)F(t, x(s), y(s))ds, t \in [0,T],$$
$$By(t) + \partial\phi(y(t)) \ni h(t, x(t), y(t)), \forall z \in Y,\ \textit{a.e.}\ t \in (0,T).$$

We firstly are concerned with the elliptic variational inequality (1.2). Consider the $EVI(g)$ problem: find $y \in X$ with given $g \in Y^*$ satisfying

$$By + \partial\phi(y) \ni g. \tag{2.2}$$

We recall a remarkable result which can be seen in [6] or in [7].

Lemma 2.1. *If* B *satisfies* (**B**) *and* $g \in X^*$*, then the solution of* (2.2) *is unique. Moreover, the corresponding*

$$\begin{aligned} \mathbb{S} :\ \ & Y^* \to Y, \\ & g \mapsto y, \end{aligned}$$

is Lipschitzian.

Proof. By [6, Theorem 2.3], we obtain that the solution of (2.2) is unique. In order to prove the map $g \to y$ is Lipschitz continuous from Y^* to Y, let y_1, y_2 be the solution of elliptic variational inequalities with respect to given data g_1, g_2, namely,

$$By_1 + \partial\phi(y_1) \ni g_1,$$
$$By_2 + \partial\phi(y_2) \ni g_2,$$

or equivalent to

$$b(y_1, y_1 - v) + \phi(y_1) - \phi(v) \leq \langle y_1 - v, g_1 \rangle, \ \forall v \in Y, \tag{2.3}$$
$$b(y_2, y_2 - v) + \phi(y_2) - \phi(v) \leq \langle y_2 - v, g_2 \rangle, \ \forall v \in Y. \tag{2.4}$$

Taking $v = y_2$ in (2.3) and $v = y_1$ in (2.4), and combining them, we have

$$b(y_1 - y_2, y_1 - y_2) \leq \langle y_1 - y_2, g_1 - g_2 \rangle.$$

Hence,

$$\|y_1 - y_2\|_Y \leq \frac{1}{\eta_B} \|g_1 - g_2\|_*,$$

or

$$\|\mathbb{S}(g_1) - \mathbb{S}(g_2)\|_Y \leq \frac{1}{\eta_B} \|g_1 - g_2\|_*, \tag{2.5}$$

thanks to (**B**), the lemma is proved. ☐

Now, for a fixed $(\tau, x) \in \mathbb{R}^+ \times X$, consider the original form of (1.2)

$$By + \partial\phi(y) \ni h(\tau, x, y). \tag{2.6}$$

Using the last lemma, we obtain the following existence result and property of solution map for (2.6).

Lemma 2.2. *Let* (**B**) *and* (**H**) *hold. In addition, suppose that* $\eta_B > \eta_{2h}$*. Then for each* $(\tau, x) \in \mathbb{R}^+ \times X$*, there exists a unique solution* $y \in Y$ *of* (2.6)*. Moreover, the solution mapping*

$$\begin{aligned} \mathbb{VI} : [0, \infty) \times X &\to Y, \\ (\tau, x) &\mapsto y, \end{aligned}$$

is Lipchizian, more precisely

$$\|\mathbb{VI}(\tau, x_1) - \mathbb{VI}(\tau, x_2)\|_Y \leq \frac{\eta_{1h}}{\eta_B - \eta_{2h}} \|x_1 - x_2\|_X. \tag{2.7}$$

Proof. Let $(\tau, x) \in \mathbb{R}^+ \times X$. We consider the map $\mathbb{S} \circ h(\tau, x, \cdot) : Y \to Y$. Employing (2.5), we have

$$\begin{aligned} \|\mathbb{S}(h(\tau, x, y_1)) - \mathbb{S}(h(\tau, x, y_2))\|_Y &\leq \frac{1}{\eta_B} \|h(\tau, x, y_1) - h(\tau, x, y_2)\|_* \\ &\leq \frac{\eta_{2h}}{\eta_B} \|y_1 - y_2\|_Y. \end{aligned}$$

Because $\eta_{2h} < \eta_B$, $y \mapsto \mathbb{S}(h(\tau, x, \cdot))$ is a contraction map, then it admits a unique fixed point, which is the unique solution of (2.6).

It remains to show the map $(\tau, x) \mapsto y$ is a Lipschitz corresponding with respect to the second variable. Let $\mathbb{VI}(\tau, x_1) = y_1, \mathbb{VI}(\tau, x_2) = y_2$. Then, one has

$$\begin{aligned}\|y_1 - y_2\|_Y &= \|\mathbb{S}(h(\tau, x_1, y_1)) - \mathbb{S}(h(\tau, x_2, y_2))\|_Y \\ &\le \frac{1}{\eta_B}\|h(\tau, x_1, y_1) - h(\tau, x_2, y_2)\|_* \\ &\le \frac{\eta_{1h}}{\eta_B}\|x_1 - x_2\|_X + \frac{\eta_{2h}}{\eta_B}\|y_1 - y_2\|_Y.\end{aligned}$$

Therefore

$$\|y_1 - y_2\|_Y \le \frac{\eta_{1h}}{\eta_B - \eta_{2h}}\|x_1 - x_2\|_X,$$

which leads to the conclusion of lemma. □

In order to solve (1.1)-(1.2), we convert it to a differential equation. We consider the following map:

$$G(t, x) := F(t, x, \mathbb{VI}(t, x)),\ (t, x) \in \mathbb{R}^+ \times X.$$

One sees that $G : \mathbb{R}^+ \times X \to X$. Moreover, by assumption **(F)** and the continuity of $\mathbb{VI}$, we observe that the map $G(t, \cdot)$ is continuous for each $t \ge 0$. By the estimate (2.7), and the Hausdorff MNC property, one has

$$\chi_Y(\mathbb{VI}(t, \Omega)) \le \frac{\eta_{1h}}{\eta_B - \eta_{2h}}\chi_X(\Omega),$$

where χ_Y is the Hausdorff MNC in Y. In the case the semigroup $S(\cdot)$ is non-compact, we have

$$\begin{aligned}\chi_X(G(t, \Omega)) &= \chi_X(F(t, \Omega, \mathbb{VI}(t, \Omega))) \\ &\le a(t)\chi_X(\Omega) + b(t)\chi_Y(\mathbb{VI}(t, \Omega)) \\ &\le a(t)\chi_X(\Omega) + b(t)\left(\frac{\eta_{1h}}{\eta_B - \eta_{2h}}\chi_X(\Omega)\right) \\ &\le \left(a(t) + \frac{b(t)\eta_{1h}}{\eta_B - \eta_{2h}}\right)\chi_X(\Omega) \\ &= p_G(t)\chi_X(\Omega),\end{aligned}$$

where $p_G(t) = \left(a(t) + \dfrac{b(t)\eta_{1h}}{\eta_B - \eta_{2h}}\right)$.

Concerning the growth of G, by **(F2)** we arrive at

$$\begin{aligned}\|G(t, x)\|_X &\le a(t)\|x\|_X + b(t)\|\mathbb{VI}(t, x)\|_Y + \|F(t, 0, 0)\|_X \\ &\le a(t)\|x\|_X + b(t)\frac{\eta_{1h}}{\eta_B - \eta_{2h}}\|x\|_X + \|\mathbb{VI}(t, 0)\|_Y + \|F(t, 0, 0)\|_X.\end{aligned}$$

By a process similar to that in Lemma 2.2, we obtain

$$\|\mathbb{VI}(t,x)\| \le \frac{\eta_h(t,0)}{\eta_B-\eta_{2h}} + \frac{\eta_{1h}}{\eta_B-\eta_{2h}}\|x\| + \|\mathbb{VI}(0,0)\|.$$

Thus, we have

$$\|G(t,x)\|_X \le \eta_G(t)\|x\|_X + d(t),$$

where $\eta_G(t) := \left(a(t) + \dfrac{b(t)\eta_{1h}}{\eta_B-\eta_{2h}}\right)$ and $d(t) = \frac{\eta_h(t,0)}{\eta_B-\eta_{2h}} + \|\mathbb{VI}(0,0)\| + \|F(t,0,0)\|_X$. In addition, we also get that

$$\begin{aligned}
\|G(t,x)-G(t,x')\|_X &= \|F(t,x,\mathbb{VI}(t,x)) - F(t,x',\mathbb{VI}(t,x'))\|_X \\
&\le a(t)\|x-x'\|_X + b(t)\|\mathbb{VI}(t,x)-\mathbb{VI}(t,x')\|_Y \\
&\le a(t)\|x-x'\|_X + \frac{b(t)\eta_{1h}}{\eta_B-\eta_{2h}}\|x-x'\|_X \\
&\le \left(a(t) + \frac{b(t)\eta_{1h}}{\eta_B-\eta_{2h}}\right)\|x-x'\|_X \\
&\le \gamma(t)\|x-x'\|_X, \qquad (2.8)
\end{aligned}$$

where $\gamma(t) = \left(a(t) + \dfrac{b(t)\eta_{1h}}{\eta_B-\eta_{2h}}\right)$.

Due to the aforementioned setting, the problem (1.1)-(1.2) is converted to

$$x'(t) - Ax(t) = G(t,x(t)), t \in [0,T],$$

Now we see that, a pair of functions (x,y) is a mild solution of (1.1)-(1.2) with initial value $x(0) = x_0$ iff

$$x(t) = S(t)x_0 + \int_0^t S(t-s)G(s,x(s))ds, t \in [0,T], \qquad (2.9)$$

$$y(t) = \mathbb{VI}(t,x(t)). \qquad (2.10)$$

Consider the *Cauchy operator*

$$\mathcal{W} : L^1(0,T,X) \to C([0,T];X),$$

$$\mathcal{W}(f)(t) = \int_0^t S(t-s)f(s)ds.$$

For a given $x_0 \in X$, we introduce the mild solution operator

$$\mathcal{F} : C([0,T];X) \to C([0,T];X),$$

$$\mathcal{F}(x) = S(\cdot)x_0 + \mathcal{W}(G(\cdot,x(\cdot))).$$

It is evident that x is a fixed point of $\mathcal{F}$ iff x is the first component of solution of (1.1)-(1.2)-(2.1). In order to prove the existence result for problem (1.1)-(1.2)-(2.1), we make use of the Schauder fixed point theorem.

Lemma 2.3. *Let E be a Banach space and $D \subset E$ be a nonempty compact convex subset. If the map $\mathcal{F} : D \to D$ is continuous, then $\mathcal{F}$ has a fixed point.*

We have the following result related to the operator $\mathcal{W}$.

Proposition 2.1. *Let* (**A**) *hold. If $D \subset L^1(0,T;X)$ is semicompact, then $\mathcal{W}(D)$ is relatively compact in $C(J;X)$. In particular, if sequence $\{f_n\}$ is semicompact and $f_n \rightharpoonup f^*$ in $L^1(0,T;X)$ then $\mathcal{W}(f_n) \to \mathcal{W}(f^*)$ in $C([0,T];X)$.*

Theorem 2.1. *Let the hypotheses* (**A**), (**B**), (**F**) *and* (**H**) *hold. Then the problem* (1.1)-(1.2)-(2.1) *has at least one mild solution $(x(\cdot), y(\cdot))$ for given $x_0 \in X$.*

Proof. We now show that there exists a nonempty convex subset $\mathcal{M}_0 \subset C([0,T];X)$ such that $\mathcal{F}(\mathcal{M}_0) \subset \mathcal{M}_0$.

Let $z = F(x)$, then we have

$$\begin{aligned}
\|z(t)\|_X &\le \|S(t)x_0\|_X + \|\int_0^t S(t-s)G(s,x(s))ds\|_X \\
&\le M\|x_0\|_X + \int_0^t \|S(t-s)\|_{\mathcal{L}(X)}\|\|G(s,x(s))\|_X ds \\
&\le M\|x_0\|_X + M\int_0^t (\eta_G(s)\|x(s)\|_X + d(s))ds,
\end{aligned}$$

where $M = \sup\{\|S(t)\|_{\mathcal{L}(X)} : t \in [0,T]\}$.

Denote

$$\mathcal{M}_0 = \{x \in C([0,T];X) : \|x(t)\|_X \le \kappa(t), \forall t \in [0,T]\},$$

where κ is the unique solution of the integral equation

$$\kappa(t) = M\|x_0\|_X + M\int_0^t (\eta_G(s)\kappa(s) + d(s))ds.$$

It is obvious that $\mathcal{M}_0$ is a closed, convex subset of $C([0,T];X)$ and $\mathcal{F}(\mathcal{M}_0) \subset \mathcal{M}_0$.

Set

$$\mathcal{M}_{k+1} = \overline{co}\mathcal{F}(\mathcal{M}_k), k = 0, 1, 2, \ldots$$

here, the notation $\overline{co}$ stands for the closure of convex hull of a subset in $C([0,T];X)$. We see that $\mathcal{M}_k$ is a closed convex set and $\mathcal{M}_{k+1} \subset \mathcal{M}_k$ for all $k \in \mathbb{N}$.

Let $\mathcal{M} = \bigcap\limits_{k=0}^{\infty} \mathcal{M}_k$, then $\mathcal{M}$ is a closed convex subset of $C([0,T];X)$ and $\mathcal{F}(\mathcal{M}) \subset \mathcal{M}$.

On the other hand, for each $k \ge 0, \mathcal{P}_G(\mathcal{M}_k)$ is integrably bounded by the growth of G. Thus, $\mathcal{M}$ is also integrably bounded.

In the sequel, we prove that $\mathcal{M}(t)$ is relatively compact for each $t \geq 0$. By the regularity of Hausdorff MNC, this will be done if $\mu_k(t) = \chi_X(\mathcal{M}_k(t)) \to 0$ as $k \to \infty$.

If $\{S(t)\}$ is a compact semigroup, we get $\mu_k(t) = 0, \forall t \geq 0$.
On the other hand, if $\{S(t)\}$ is noncompact, we have

$$\begin{aligned}\mu_{k+1}(t) &\leq \chi_X(\int_0^t S(t-s)G(s, \mathcal{M}_k(s))ds) \\ &\leq 4M \int_0^t \chi_X(G(s, \mathcal{M}_k(s)))ds \\ &\leq 4M \int_0^t p_G(s)\chi(\mathcal{M}_k(s))ds.\end{aligned}$$

Hence,

$$\mu_{k+1}(t) \leq 4M \int_0^t p_G(s)\mu_k(s)ds.$$

Putting $\mu_\infty(t) = \lim\limits_{k\to\infty} \mu_k(t)$ and passing to the limit we have

$$\mu_\infty(t) \leq 4M \int_0^t p_G(s)\mu_\infty(s)ds.$$

By using the Gronwall inequality, we obtain $\mu_\infty(t) = 0$ for all $t \in J$. Hence, $\mathcal{M}(t)$ is relatively compact for all $t \in J$.

By Proposition 2.1, $\mathcal{W}(\mathcal{M})$ is relatively compact in $C([0,T];X)$. Then $\mathcal{F}(\mathcal{M})$ is a relatively compact subset in $C([0,T];X)$.

Let us put

$$D = \overline{co}\Phi(\mathcal{M}).$$

It is easy to see that D is a nonempty compact convex subset of $C([0,T];X)$ and $\mathcal{F}(D) \subset D$ because $\mathcal{F}(D) = \mathcal{F}(\overline{co}\mathcal{F}(\mathcal{M})) \subset \mathcal{F}(\mathcal{M}) \subset \overline{co}\mathcal{F}(\mathcal{M}) = D$.

We now consider $\mathcal{F} : D \to D$. In order to apply the fixed point principle given by Lemma 2.3, it remains to show that $\mathcal{F}$ is a continuous map. Let $x_n \in D$ with $x_n \to x^*$ and $y_n \in \mathcal{F}(x_n)$ with $y_n \to y^*$. Then $y_n(t) = S(t)x_0 + \int_0^t S(t-s)G(s, x_n(s))ds$. By the continuity of G we can pass to the limit to get that

$$x^*(t) = S(t)x_0 + \int_0^t S(t-s)G(s, x^*(s))ds.$$

Then $\mathcal{F}$ has a fixed point $\mathbf{x}$. Therefore, let $\mathbf{y}(\cdot) = \mathbb{VI}(\cdot, \mathbf{x}(\cdot))$, we conclude that $(\mathbf{x}, \mathbf{y})$ is a mild solution of our problem. $\square$

Theorem 2.2. *Under the assumptions* (**A**), (**B**), (**F**) *and* (**H**), *the system* (1.1)-(1.2) *has a unique mild solution for each initial value* $x(0) = x_0$.

Proof. Let (x_1, y_1) and (x_2, y_2) be two mild solutions of (1.1)-(1.2) such that $x_1(0) = x_2(0) = x_0$, we have

$$x_1(t) = S(t)x_0 + \int_0^t S(t-s)G(s, x_1(s))ds,$$

$$x_2(t) = S(t)x_0 + \int_0^t S(t-s)G(s, x_2(s))ds.$$

Then subtracting two last equations, we have

$$x_1(t) - x_2(t) = \int_0^t S(t-s)(G(s, x_1(s)) - G(s, x_2(s)))ds.$$

By estimate of G, we obtain that

$$\begin{aligned}\|x_1(t) - x_2(t)\|_X &\leq \int_0^t \|S(t-s)\|_{\mathcal{L}(X)} \|G(s, x_1(s)) - G(s, x_2(s))\|_X ds \\ &\leq M \int_0^t \gamma(s)\|x_1(s) - x_2(s)\|_X ds.\end{aligned}$$

Using the Gronwall inequality, we deduce the uniqueness of mild solution. □

2.2. The existence of mild periodic solution

In this section, let $T > 0$ be a positive time. We replace **(A)**, **(F)**, **(H)** by the following assumptions:

($\mathbf{A}^*$) *A satisties* (**A**) *and the semigroup $S(t)$ is is exponentially stable with exponent α, that is*

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{-\alpha t}, \forall t > 0.$$

($\mathbf{F}^*$) *F satisfies* (**F**) *with $a(t) \equiv a$ and $b(t) \equiv b$. Moreover,*

$$F(t, x, y) = F(t+T, x, y),\ \forall t \geq 0, x \in X, y \in Y;$$

($\mathbf{H}^*$) *h satisfies* **(H)** *and*

$$h(t, x, y) = h(t+T, x, y)\ \forall t \geq 0, x \in X, y \in Y.$$

Definition 2.2. *A pair of continuous functions (x, y) is called a mild T-periodic solution of* (1.1)-(1.2) *iff*

$$\begin{aligned}&x(t) = S(t-s)x(s) + \int_s^t S(t-s)F(s, x(s), y(s))ds,\ \forall t \geq s \geq 0,\\ &x(t) = x(t+T),\ \forall t \geq 0,\\ &By(t) + \partial(\phi(y(t))) \ni h(t, x(t), y(t)),\ \textit{ for a.e. } t \geq 0.\end{aligned}$$

By Theorem 2.2, due to the unique solvability of (2.9)-(2.10), we define the following map:

$\mathcal{G}: X \to X,$

$\mathcal{G}(x_0) = S(T)x_0 + \int_0^T S(T-s)G(s,x(s))ds,$ where x is a mild solution of (2.9) with $x(0) = x_0.$

The following theorem shows the main result of this section.

Theorem 2.3. *Under the assumptions* $(\mathbf{A}^*), (\mathbf{B}), (\mathbf{F}^*)$ *and* $(\mathbf{H}^*)$*, the system* (1.1)-(1.2) *has a unique mild T-periodic solution, provided that $\eta_B > \eta_{2h}$ and the estimates hold*

$$\alpha > M(a + \frac{b\eta_{1h}}{\eta_B - \eta_{2h}}), \tag{2.11}$$

$$M \exp\left(-\left(\alpha - M(a + \frac{b\eta_{1h}}{\eta_B - \eta_{2h}})\right)T\right) < 1. \tag{2.12}$$

Proof. First of all, we prove that $\mathcal{G}$ has a fixed point. For any $\xi_1, \xi_2 \in X$, let $x_1 = x_1(\cdot;\xi_1), x_2 = x_2(\cdot;\xi_2)$ be the mild solutions of (2.9) with initial values ξ_1, ξ_2, respectively. We have

$$\mathcal{G}(\xi_1) - \mathcal{G}(\xi_2) = S(T)(\xi_1 - \xi_2) + \int_0^T S(T-s)(G(s,x_1(s)) - G(s,x_2(s)))ds.$$

By the integral formula of mild solution, one has

$$x_1(t) - x_2(t) = S(t)(\xi_1 - \xi_2) + \int_0^t S(t-s)(G(s,x_1(s)) - G(s,x_2(s)))ds.$$

Then employing (2.8), we get

$$\begin{aligned}\|x_1(t) - x_2(t)\|_X &\le \|S(t)\|_{\mathcal{L}(X)}\|\xi_1 - \xi_2\|_X + \int_0^t \|S(t-s)\|_{\mathcal{L}(X)}\|G(s,x_1(s)) - G(s,x_2(s))\|_X ds \\ &\le Me^{-\alpha t}\|\xi_1 - \xi_2\|_X + M\int_0^t e^{-\alpha(t-s)}\gamma\|x_1(s) - x_2(s)\|_X ds,\end{aligned}$$

where $\gamma = a + \frac{b\eta_{1h}}{\eta_B - \eta_{2h}}$. Hence,

$$e^{\alpha t}\|x_1(t) - x_2(t)\|_X \le M\|\xi_1 - \xi_2\|_X + M\gamma\int_0^t e^{\alpha s}\|x_1(s) - x_2(s)\|_X ds.$$

Using the Gronwall inequality, we have

$$e^{\alpha t}\|x_1(t) - x_2(t)\|_X \le M\|\xi_1 - \xi_2\|_X e^{M\gamma t}.$$

Then,

$$\|x_1(t) - x_2(t)\|_X \le M\|\xi_1 - \xi_2\|_X e^{-(\alpha - M\gamma)t}.$$

From then, one has

$$\begin{aligned}
\|\mathcal{G}(\xi_1) - \mathcal{G}(\xi_2)\|_X &\le Me^{-\alpha T}\|\xi_1 - \xi_2\|_X + \int_0^T Me^{-\alpha(T-s)}\gamma\|x_1(s) - x_2(s)\|_X ds \\
&\le Me^{-\alpha T}\|\xi_1 - \xi_2\|_X + \int_0^T Me^{-\alpha(T-s)}\gamma M\|\xi_1 - \xi_2\|_X e^{-(\alpha - M\gamma)s} ds \\
&= Me^{-(\alpha - M\gamma)T}\|\xi_1 - \xi_2\|_X.
\end{aligned}$$

Then, by the estimations (2.11)-(2.12), it implies that $\mathcal{G}$ has a unique fixed point in X. We suppose that $\mathcal{G}(x^*) = x^*$. By the definition of $\mathcal{G}$, there exists a unique mild solution $\bar{x}(t)$ satisfying

$$\bar{x}(t) = S(t)x^* + \int_0^t S(t-s)G(s, \bar{x}(s))ds,$$

and $\bar{x}(0) = \bar{x}(T) = x^*$. This fixed point is the initial value from which the mild T-periodic solution starts. Then, define $\bar{\mathbf{x}}(t)$ by

$$\bar{\mathbf{x}}(t) = \bar{x}(t - kT),\ t \in [kT, (k+1)T],\ k = 0, 1, 2, ...$$

and we define

$$\bar{\mathbf{y}}(t) = \mathbb{VI}(t, \bar{\mathbf{x}}(t)),\ t \ge 0,$$

which yields that $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is mild periodic solution of (1.1)-(1.2). $\square$

3. Application

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the following problem

$$\begin{aligned}
&\frac{\partial Z}{\partial t}(t, x) - \Delta_x Z(t, x) = f(t, x, Z(t, x), u(t, x)), &(3.1)\\
&-\Delta_x u(t, x) + \beta(u(t, x) - \psi(x)) \ni h(t, x, Z(t, x), u(t, x)), &(3.2)\\
&Z(t, x) = 0, u(t, x) = 0,\ x \in \partial\Omega, t \ge 0, &(3.3)
\end{aligned}$$

with the periodic condition

$$Z(t, x) = Z(t + T, x),\ \forall x \in \Omega, t \in \mathbb{R}^+,$$

where $T > 0$. The maps $f, h : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions, ψ is in $H^2(\Omega)$ and $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone graph

$$\beta(r) = \begin{cases} 0 & \text{if } r > 0, \\ \mathbb{R}^- & \text{if } r = 0, \\ \emptyset & \text{if } r < 0. \end{cases}$$

Note that, parabolic variational inequality (3.2) reads as follows:

$$
\begin{aligned}
-\Delta_x u(t,x) &= h(x, Z(t,x)) \ \text{ in } \ \{(t,x) \in Q := (0,T) \times \Omega : u(t,x) \geq \psi(x)\},\\
-\Delta_x u(t,x) &\geq h(x, Z(t,x)), \ \text{ in } \ Q,\\
u(t,x) &\geq \psi(x), \forall (t,x) \in Q,
\end{aligned}
$$

which represents a rigorous and efficient way to treat dynamic diffusion problems with a free or moving boundary. This model is called the *obstacle parabolic problem* (see [6]).

Let $X = L^2(\Omega), Y = H_0^1(\Omega)$, the norm in X and Y is given by

$$|u| = \sqrt{\int_\Omega u^2(x)dx}, u \in L^2(\Omega).$$

The norm in $H_0^1(\Omega)$ is given by

$$\|u\| = \sqrt{\int_\Omega |\nabla u(x)|^2 dx}, u \in H_0^1(\Omega).$$

Define the abstract function

$$
\begin{aligned}
&F : \mathbb{R}^+ \times X \times Y \to \mathcal{P}(X)\\
&F(t, Z, u) = f(t, x, Z(x), u(x)),
\end{aligned}
$$

and the operator

$$A = \Delta : D(A) \subset X \to X; D(A) = \{H^2(\Omega) \cap H_0^1(\Omega)\}.$$

Then (3.1) can be reformulated as

$$Z'(t) - AZ(t) = F(t, Z(t), u(t)),$$

where $Z(t) \in X, u(t) \in Y$ such that $Z(t)(x) = Z(t,x)$ and $u(t)(x) = u(t,x)$. It is known that ([8]), the semigroup $S(t)$ generated by A is compact and exponentially stable, that is,

$$\|S(t)\|_{\mathcal{L}(X)} \leq e^{-\lambda_1 t},$$

then the assumption $(\mathbf{A}^*)$ is satisfied.

We assume, in addition, that there exist nonnegative functions $a(\cdot), b(\cdot) \in L^\infty(\Omega)$ such that

$$|f(t,x,p,q) - f(t,x,p',q')| \leq a(x)|p - p'| + b(x)|q - q'|,$$

and moreover, we suppose $f(t,x,p,q) = f(t+T,x,p,q)$ for all $t \geq 0, x \in \Omega, p, q \in \mathbb{R}$.

By the setting of function F, it is easy to see that F is continuous and

$$\|F(t,Z,u)-F(t,\bar{Z},\bar{u})\| \le \|a\|_\infty \|Z-\bar{Z}\|_X + \frac{\|b\|_\infty}{\sqrt{\lambda_1}}\|u-\bar{u}\|_Y.$$

Thus, **(F)** holds.

Consider the elliptic variational inequality (3.2), putting $B=-\Delta$, where $-\Delta$ is Laplace operator

$$\langle u, -\Delta v\rangle := \int_\Omega \nabla u(x)\nabla v(x)dx,$$

then $\langle Bu,u\rangle = \|u\|_U^2$. So, the assumption **(B)** is testified with $\eta_B=1$.

The map $h:\mathbb{R}^+\times\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies $h(t,x,p,q)=h(t+T,x,p,q), \forall x\in\Omega, t\ge 0, p,q\in\mathbb{R}$ and

$$|h(t,x,p,q)-h(\bar{t},x,p',q')| \le \eta(t,\bar{t}) + c(x)|p-p'| + d(x)|q-q'|, \forall x\in\Omega, p,q\in\mathbb{R},$$

where $c(\cdot), d(\cdot)$ are the nonnegative functions in $L^\infty(\Omega)$ and $\eta(\cdot,\cdot):\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a nonnegative continuous function.

Let $h:\mathbb{R}^+\times X\times Y\to L^2(\Omega), h(t,\bar{Z},\bar{u})(x)=h(t,x,\bar{Z}(x),\bar{u}(x))$, we obtain

$$|h(t,Z,u)-h(\bar{t},\bar{Z},\bar{u})| \le \|c\|_\infty\|Z-\bar{Z}\|_X + \frac{\|d\|_\infty}{\sqrt{\lambda_1}}\|u-\bar{u}\|_Y + \eta(t,\bar{t})|\Omega|.$$

Then the EVI (3.2) reads as

$$Bu(t)+\partial I_K(u(t)) \ni h(t,Z(t),u(t)),$$

where

$$\begin{aligned} K &= \{u\in H_0^1(\Omega): u(y)\ge\psi(x), \text{ for a.e. } x\in\Omega\},\\ \partial I_K(u) &= \Big\{u\in H_0^1(\Omega): \int_\Omega u(x)(v(x)-z(x))dx \ge 0, \forall z\in K\Big\},\\ &= \{u\in H_0^1(\Omega): u(x)\in\beta(v(x)-\psi(x)), \text{ for a.e. } x\in\Omega\}. \end{aligned}$$

It follows that **(H)** is testified.
We have the following result due to Theorem 2.3.

Theorem 3.1. *If* $\|d\|_\infty^2 < \lambda_1$ *and*

$$\|a\|_\infty + \frac{\|b\|_\infty\|c\|_\infty}{\sqrt{\lambda_1}-\|d\|_\infty} < \lambda_1,$$

then the problem (3.1)-(3.3) *has a unique mild* T*-periodic solution* $(\mathbf{Z},\mathbf{u})$.

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