ON THE SECOND MAIN THEOREM FOR HOLOMORPHIC CURVES INTO LINEAR PROJECTIVE SUBSPACE

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ABSTRACT

Value distribution theory for holomorphic curves which also known as Nevanlinna-Cartan theory was originated by the work of H. Cartan in 1933. Since that time, it had attracted the attention of many mathematicians and had many important publications and it had many applications in different areas of mathematics. Recently, J. M. Anderson and A. Hinkkanen introduced the integrated reduced counting functions for holomorphic curves and proved an improved version of second main theorem for holomorphic curves with integrated reduced counting functions in the complex case. Our idea here is to consider the Anderson and A. Hinkkanen's result for the case of holomorphic curves into a linear projective subspace. The main result in this paper is Main Theorem, which is an improved of second main theorem for holomorphic curves with reduced counting functions.

ĐỊNH LÝ CƠ BẢN THỨ HAI CHO ĐƯỜNG CONG CHỈNH HÌNH VÀO KHÔNG GIAN CON TUYẾN TÍNH XẠ ẢNH

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TỪ KHÓA

Đường cong chỉnh hình
Lý thuyết Nevanlinna-Cartan
Định lý cơ bản thứ hai
Không gian con
Hàm đếm rút gọn

TÓM TẮT

Lý thuyết phân bố giá trị cho đường cong chỉnh hình hay còn gọi là lý thuyết Nevanlinna-Cartan khởi nguồn bởi các công việc của H. Cartan vào năm 1933. Từ đó đến nay lý thuyết này đã nhận được sự quan tâm của nhiều nhà toán học trên thế giới và có nhiều công trình công bố quan trọng và có nhiều ứng dụng trong các lĩnh vực khác nhau của toán học. Gần đây J. M. Anderson và A. Hinkkanen giới thiệu hàm đếm rút gọn cho đường cong chỉnh hình và chứng minh một phiên bản mới của định lý cơ bản thứ hai cho đường cong chỉnh hình với hàm đếm mới trong trường hợp phức. Ý tưởng của chúng tôi ở đây là xem xét kết quả của Anderson và A. Hinkkanen cho trường hợp đường cong chỉnh hình vào một không gian con tuyến tính xạ ảnh. Kết quả chính của chúng tôi là Main Theorem, định lý này là một dạng định lý cơ bản thứ hai cho đường cong chỉnh hình với hàm đếm mới.

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1. Introduction

Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map and let $f = (f_0 : \cdots : f_n)$ be a reduced representative of f, where $f_0, \ldots, f_n$ are entire functions on $\mathbb{C}$ without common zeros. The Nevanlinna-Cartan characteristic function $T_f(r)$ is defined by

$$T_f(r) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log \|f(re^{i\theta})\| d\theta, \tag{1.1}$$

where $\|f(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}$.

Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ and let L be the linear form defining H. Let $n_f(r, H)$ be the number of zeros of $L \circ f$ in the disk $|z| < r$, counting multiplicity, and $n_f^{\Delta}(r, L)$ be the number of zeros of $L \circ f$ in the disk $|z| < r$, truncated multiplicity by a positive integer Δ. The counting function and truncated counting function are defined by

$$N_f(r, H) = \int\limits_0^r \frac{n_f(t, H) - n_f(0, H)}{t} dt + n_f(0, H) \log r; \tag{1.2}$$

$$N_f^{\Delta}(r, H) = \int\limits_0^r \frac{n_f^{\Delta}(t, H) - n_f^{\Delta}(0, H)}{t} dt + n_f^{\Delta}(0, H) \log r. \tag{1.3}$$

Let X be a k-dimensional linear projective subspace of $\mathbb{P}^n(\mathbb{C}), 1 \leqslant k \leqslant n$. A collection of hypersurfaces $\{H_1, \ldots, H_q \ (q \geq k+1)\}$ in $\mathbb{P}^n(\mathbb{C})$, which are defined by linear forms $L_j,\ 1 \leqslant j \leqslant q$, is said to be *in general position with* X if for any subset $\{i_0, \ldots, i_k\}$ of $\{1, \ldots, q\}$ of cardinality $k+1$,

$$\{x \in X : L_{i_j}(x) = 0,\ j = 0, \ldots, k\} = \emptyset. \tag{1.4}$$

When $k = n$, we call the collection of hypersurfaces $\{H_1, \ldots, H_q\}$ in general position.

Năm 1933, in [1], H. Cartan showed the following

Theorem A ([1]). *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an linearly non-degenerate holomorphic map, and let $\{H_1, \ldots, H_q\}$ be a collection of hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Then we have for any $\varepsilon > 0$,*

$$(q - n - 1 - \varepsilon)T_f(r) \leqslant \sum_{j=1}^{q} N_f^n(r, H_j) + O(1) \tag{1.5}$$

for all enough large $r > 0$, outside a set of Lebesgue finite measure.

In 1983, Nochka ([2]) established a truncated defect relation for a linearly non-degenerate holomorphic map intersecting hyperplanes. In 2004, M.Ru ([3]) established a defect relation for algebraically non-degenerate holomorphic map intersecting hypersurfaces. The other results of the value distributions theory of for holomorphic curves with counting functions can be found in [4], [5], [6], [7], [8]. In 2014, J. M. Anderson and A. Hinkkanen ([9]) improved of Cartan's result and proved a version of second main theorem for holomorphic curves with integrated reduced counting functions in the complex case. Now we introduce this result.

Let $g_0, \ldots, g_p$ be are entire functions on $\mathbb{C}$ without common zeros and linearly independent over $\mathbb{C}$, we denote by $W(g_0, \ldots, g_p)$ the Wronskian determinant of $g_0, \ldots, g_p$ and denote by $\mathcal{L}(g_0, \ldots, g_p)$ the set of all of non-trivial linear combinations of $g_0, \ldots, g_p$.

Let X be a k-dimensional linear projective subspace of $\mathbb{P}^n(\mathbb{C}), 1 \leqslant k \leqslant n$, and let $f = (f_0 : \cdots : f_n) : \mathbb{C} \to X$ be a linear non-degenerate holomorphic map, where $f_0, \ldots, f_n$ have no common zeros. Then there are $k+1$ functions $f_{s_0}, \ldots, f_{s_k}$, which are linearly independent, and f_s can be written as a linear form of $f_{s_0}, \ldots, f_{s_k}$ for any $s \notin \{s_0, \ldots, s_k\}$. We denote $W_f = W(f_{s_0}, \ldots, f_{s_k})$ the Wronskian determinant of $f_{s_0}, \ldots, f_{s_k}$. And it is easy to check from definition $\mathcal{L}(f_0, \ldots, f_n) = \mathcal{L}(f_{s_0}, \ldots, f_{s_k})$.

For any $z \in \mathbb{C}$, from Lemma 1, we have the possible orders of the zeros of the functions in $\mathcal{L}(f_{s_0}, \ldots, f_{s_k})$ form the sequence

$$\{0 = d_0(z) < d_1(z) < \cdots < d_k(z)\}. \tag{1.6}$$

The integer numbers $d_0(z), d_1(z), \ldots, d_k(z)$ are said to be the *characteristic exponents* of $f_{s_0}, \ldots, f_{s_k}$ at z. Since $\mathcal{L}(f_0, \ldots, f_n) = \mathcal{L}(f_{s_0}, \ldots, f_{s_k})$, the possible orders of the zeros of the functions in $\mathcal{L}(f_0, \ldots, f_n)$ also form the sequence $d_0(z), d_1(z), \ldots, d_k(z)$, which also are said to be the *characteristic exponents* of $f_0, \ldots, f_n$ at z. From Lemma 2, this characteristic exponents does not depend on the choice of $f_{s_0}, \ldots, f_{s_k} \in \{f_0, \ldots, f_n\}$ as long as $f_{s_0}, \ldots, f_{s_k}$ are linearly independent.

Now let H be hyperplane in $\mathbb{P}^n(\mathbb{C})$, which is defined by a linear form L, obviously $L(f) \in \mathcal{L}(f_0, \ldots, f_n) = \mathcal{L}(f_{s_0}, \ldots, f_{s_k})$. So for any $z \in \mathbb{C}$, there is an integer number $j \in \{0, 1, \ldots, k\}$ such that $d_j(z)$ is the order of $L(f)$ at z, here $d_0(z), \ldots, d_k(z)$ are the characteristic exponents of $f_0, \ldots, f_n$ at z_0. We say $\nu(H, z) = j$ the *reduced multiplicity* of zero of $L(f)$ at z and $\varepsilon(H, z) = d_j(z) - j$ is the *excess* of $L(f)$ at z. It is easy to see that

$$\nu(H, z) \leqslant \min\{d_j(z), k\}, \tag{1.7}$$

and $\varepsilon(H, z) \geqslant 0$, $\varepsilon(H, z) = 0$ when $W_f(z) \neq 0$ from Lemma 1.

We denote the new non-integrated counting function of zeros of $L(f)$ by

$$\nu_f(r, H) = \sum_{|z| \leqslant r} \nu(H, z). \tag{1.8}$$

The *integrated reduced counting function* of f is defined by

$$\mathcal{N}_f(r, H) = \int_0^r \frac{\nu_f(t, H) - \nu_f(0, H)}{t} dt + \nu_f(0, H) \log r. \tag{1.9}$$

Now let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a collection of $q \geq k+1$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$ and let L_j is the linear form defining H_j for $j = 1, 2, \ldots, q$. We set

$$H = \frac{L_1(f) L_2(f) \ldots L_q(f)}{W_f}. \tag{1.10}$$

And for any $z \in \mathbb{C}$, we set

$$\mathcal{V}(\mathcal{H}, z) = \mathrm{ord}_{W_f}(z) - \sum_{j=1}^{q} \varepsilon(H_j, z), \tag{1.11}$$

here $\mathrm{ord}_{W_f}(z)$ is order of W_f at z. It is easy to see that if W_f has a zero of order $m \geqslant 1$ at $z \in \mathbb{C}$, then $\mathcal{V}(\mathcal{H}, z) \geqslant 0$ by Lemma 3 and if $W_f(z) \neq 0$ then $\mathcal{V}(\mathcal{H}, z) = 0$ by Lemma 2.

For $r \geqslant 0$, we set $\mathcal{V}_f(r, \mathcal{H}) = \sum_{|z|\leqslant r} \mathcal{V}(\mathcal{H}, z)$ and call

$$\mathcal{U}_f(r, \mathcal{H}) = \int_0^r \frac{\mathcal{V}_f(t, \mathcal{H}) - \mathcal{V}_f(0, \mathcal{H})}{t} dt - \mathcal{V}_f(0, \mathcal{H}) \log r \tag{1.12}$$

the *counting function of the unrealized excesses for* $\mathcal{H}$.

In the case of $k = n$, năm 2014, J. M. Anderson and A. Hinkkanen showed

Theorem B ([9]). *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve, and let $\mathcal{H} = \{H_1, \dots, H_q\}$ be a collection of $q \geqslant n+1$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Then we have*

$$\begin{aligned}(q - n - 1)T_f(r) \leqslant \sum_{j=1}^{q} \mathcal{N}_f(r, H_j) - \mathcal{U}_f(r, \mathcal{H}) - N(r, H)\\ + O(\log r) + O(\log T_f(f)),\end{aligned} \tag{1.13}$$

as $r \to \infty$ outside a set of finite linear measure.

In this paper, we will prove an improved version of Theorem B in the case of f is holomorphic curve into a linear projective subspace of $\mathbb{P}^n(\mathbb{C})$. Our result is stated as follows:

Main Theorem. *Let X be a k-dimension linear projective subspace of $\mathbb{P}^n(\mathbb{C})$ and let $f : \mathbb{C} \to X$ be a linearly non-degenerate holomorphic map. Let $\mathcal{H} = \{H_1, \dots, H_q\}$ be a collection of $q \geqslant k+1$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position with X. Then we have*

$$\begin{aligned}(q - k - 1)T_f(r) \leqslant \sum_{j=1}^{q} \mathcal{N}_f(r, H_j) - \mathcal{U}_f(r, \mathcal{H}) - N(r, H)\\ + O(\log r) + O(\log T_f(r)),\end{aligned} \tag{1.14}$$

as $r \to \infty$ outside a set of finite linear measure.

Note that, when $X = \mathbb{P}^n(\mathbb{C})$ then $k = n$, $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ is a linearly non-degenerate holomorphic map and hyperplanes $H_j,\ j = 1, \dots, q$ are in general position in $\mathbb{P}^n(\mathbb{C})$. Hence Theorem B is a special case of Main Theorem when $k = n$.

2. Some Preparations

Let $f_0, \dots, f_p$ are entire functions on $\mathbb{C}$ without common zeros and linearly independent over $\mathbb{C}$. Set $W = W(f_0, \dots, f_p)$ is wronskian of the functions $f_0, \dots, f_p$. And let $\mathcal{L}(f_0, \dots, f_p)$ be the set of all of non-trivial linear combinations of $f_0, \dots, f_n$. In [9], Anderson and Hinkkanen showed a relationship between the wronskian of $f_0, \dots, f_p$

and the possible orders of zeros of functions in $\mathcal{L}(f_0, \ldots, f_p)$ in the complex case as followings:

Lemma 1 ([9]). *For $z_0 \in \mathbb{C}$, the possible orders of zeros of functions in $\mathcal{L}(f_0, \ldots, f_p)$ at z_0 form the sequence $\{d_0(z_0), d_1(z_0), \ldots, d_p(z_0)\}$ such that*

i) If $W(z_0) \neq 0$ then $d_0(z_0) = 0 < d_1(z_0) < \cdots < d_p(z_0) = p$;

ii) If $W(z_0) = 0$ then $d_0(z_0) = 0 < d_1(z_0) < \cdots < d_p(z_0)$ depend on z_0, furthermore the order of the zero of W at z_0 is equal to

$$\sum_{j=1}^{p} d_j - \frac{p(p+1)}{2}.$$

Lemma 2. *Let X be a k-dimension linear projective subspace of $\mathbb{P}^n(\mathbb{C})$ and let $f = (f_0 : \cdots : f_n) : \mathbb{C} \to X$ be a linearly non-degenerate holomorphic map. Assume that $f_{s_0}, \ldots, f_{s_k}$ and $f_{t_0}, \ldots, f_{t_k}$ are two subset of $\{f_0, \ldots, f_n\}$, which are linearly independent. Let $d_0(z), d_1(z), \ldots, d_k(z)$ are characteristic exponents of the functions $f_{s_0}, \ldots, f_{s_k}$ at z and $t_0(z), t_1(z), \ldots, t_k(z)$ are characteristic exponents of the functions $f_{t_0}, \ldots, f_{t_k}$ at z. Then we have*

$$W(f_{s_0}, \ldots, f_{s_k}) = C.W(f_{t_0}, \ldots, f_{t_k}),$$

where C is a non-zero constant, and

$$\{d_0(z), d_1(z), \ldots, d_k(z)\} = \{t_0(z), t_1(z), \ldots, t_k(z)\}.$$

Proof. Since f is a linearly non-degenerate holomorphic map, we have the functions f_s can be written as a linear form of $f_{s_0}, \ldots, f_{s_k}$ for any $s \notin \{s_0, \ldots, s_k\}$ and the functions f_t can be written as a linear form of $f_{t_0}, \ldots, f_{t_k}$ for any $s \notin \{t_0, \ldots, t_k\}$. Obviously

$$\mathcal{L}(f_{s_0}, \ldots, f_{s_k}) = \mathcal{L}(f_0, \ldots, f_n) = \mathcal{L}(f_{t_0}, \ldots, f_{t_k}).$$

This implies that from properties of Wronskian

$$W(f_{s_0}, \ldots, f_{s_k}) = C.W(f_{t_0}, \ldots, f_{t_k}),$$

here C is a non-zero constant.

Now we prove $t_j(z_0) \in \{d_0(z), d_1(z), \ldots, d_k(z)\}$ for any $j \in \{0, \ldots, k\}$. Indeed since $t_j(z)$ is characteristic exponent the functions $f_{t_0}, \ldots, f_{t_k}$ at z_0, there is a $g(z) \in \mathcal{L}(f_{t_0}, \ldots, f_{t_k})$ such that

$$\mathrm{ord}_g(z) = t_j(z).$$

Since

$$\mathcal{L}(f_{s_0}, \ldots, f_{s_k}) = \mathcal{L}(f_{t_0}, \ldots, f_{t_k}),$$

we have $g(z) \in \mathcal{L}(f_{s_0}, \ldots, f_{s_k})$, so $\mathrm{ord}_g(z) \in \{d_0(z), d_1(z), \ldots, d_k(z)\}$. This implies that $t_j(z) \in \{d_0(z), d_1(z), \ldots, d_k(z)\}$.

Similarly, we have $d_j(z) \in \{t_0(z), t_1(z), \ldots, t_k(z)\}$ for any $j \in \{0, \ldots, k\}$. This implies that $\{d_0(z), d_1(z), \ldots, d_k(z)\} = \{t_0(z), t_1(z), \ldots, t_k(z)\}$. □

Lemma 3 ([9]). *Let $f = (f_0, \ldots, f_p) : \mathbb{C} \to \mathbb{P}^p(\mathbb{C})$ be a linearly non-degenerate holomorphic map and let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a collection of $q \geqslant p+1$ hyperplanes in $\mathbb{P}^p(\mathbb{C})$ in general position. Assuming that the Wronskian of $f_0, \ldots, f_p$ has a zero of order $m \geqslant 1$ at $z_0 \in \mathbb{C}$, then*

$$\sum_{j=1}^{q} \varepsilon(H_j, z_0) \leqslant m.$$

3. Proof of Main Theorem

Let $(f_0 : \cdots : f_n)$ be a reduced representative of f, where $f_0, \ldots, f_n$ are entire functions have no common zeros. Since f is a linear non-degenerate holomorphic map into a $k-$dimension linear projective subspace, there are $(k+1)$ functions $f_{s_0}, \ldots, f_{s_k}$, which are linearly independent, and $f_s,\ s \notin \{s_0, \ldots, s_k\}$, can be written as a linear form of $f_{s_0}, \ldots, f_{s_k}$.

Without loss of generality, we may assume (by rearranging the indices $\{0, \ldots, n\}$) that $f_0, \ldots, f_k$ are linearly independent, and

$$f_s = \sum_{i=0}^{k} b_{s,i} f_i,\ s = k+1, \ldots, n.$$

Set $f^* = (f_0 : \cdots : f_k) : \mathbb{C} \to \mathbb{P}^k(\mathbb{C})$, so we have f^* is a linear non-degenerate holomorphic map on $\mathbb{P}^k(\mathbb{C})$. And set

$$W_f = W(f_0, \ldots, f_k).$$

Now let $L_j,\ j = 1, \ldots, q$, be the linear forms in $\mathbb{C}[z_0, \ldots, z_n]$ defining L_j. For any $j = 1, \ldots, q$, we set

$$L_j^* = L_j^*(z_0, \ldots, z_k) = L_j\left(z_0, \ldots, z_k, \sum_{i=0}^{k} b_{k+1,i} z_i, \ldots, \sum_{i=0}^{k} b_{n,i} z_i\right).$$

Then L_j^* is a linear form in $\mathbb{C}[z_0, \ldots, z_k]$. Let H_j^* be the hyperplane in $\mathbb{P}^k(\mathbb{C})$ which is defined by the linear form L_j^* for $j = 1, \ldots, q$. Next we show that the hyperplanes $H_j^*,\ j = 1, \ldots, q$, are in general position with $\mathbb{P}^k(\mathbb{C})$. Assume for the sake contradiction that there are $(k+1)$ hyperplanes $H_{i_0}^*, \ldots, H_{i_k}^* \in \{H_1^*, \ldots, H_q^*\}$ and $\mathbf{a}^* = (a_0, \ldots, a_k) \in \mathbb{P}^k(\mathbb{C})$ such that

$$L_{i_0}^*(\mathbf{a}^*) = \cdots = L_{i_k}^*(\mathbf{a}^*) = 0.$$

Set

$$\mathbf{a} = \left(a_0, \ldots, a_k, \sum_{i=0}^{k} b_{k+1,i} a_i, \ldots, \sum_{i=0}^{k} b_{n,i} a_i\right),$$

then $\mathbf{a} \in X$ and

$$L_{i_0}(\mathbf{a}) = \cdots = L_{i_k}(\mathbf{a}) = 0.$$

This is a contradiction with the assumption "in general position with X" of hyperplanes $H_j,\ j = 1, \ldots, q$.

Set

$$H^*(z) = \frac{L_1^*(f)L_2^*(f)\dots L_q^*(f)}{W_f}.$$

Applying Theorem B to the linearly non-degenerate holomorphic map $f^* : \mathbb{C} \to \mathbb{P}^k(\mathbb{C})$ and collection of hyperplanes $\mathcal{H}^* = \{H_j^*,\ j = 1, \dots, q\}$ we have

$$\begin{aligned}(q-k-1)T_{f^*}(r) \leqslant \sum_{j=1}^{q} \mathcal{N}_{f^*}(r, H_j^*) - \mathcal{U}_{f^*}(r, \mathcal{H}^*)\\ - N(r, H^*) + O(\log r) + O(\log T_{f^*}(r))\end{aligned} \tag{3.1}$$

where inequality (3.1) holds for all large positive real number r.

We now estimate both sides of the above inequality. For $z \in \mathbb{C}$ and for any $s = k+1, \dots, n$ we have

$$\begin{aligned}|f_s(z)| = |\sum_{i=0}^{k} b_{s,i} f_i(z)| \leqslant \sum_{i=0}^{k} |b_{s,i} f_i(z)| \leqslant \sum_{i=0}^{k} |b_{s,i}|.|f_i(z)|\\ \leqslant \max\{|f_0(z)|, \dots, |f_k(z)|\}. \sum_{i=0}^{k} |b_{s,i}| = c_s. \max\{|f_0(z)|, \dots, |f_k(z)|\}.\end{aligned}$$

where c_s is a positive constant, depends only on the $b_{s,i}$ and not on z and f^*. Set

$$c = \max\{1, c_{k+1}, \dots, c_n\},$$

then we have, for any $z \in \mathbb{C}$,

$$|f_s(z)| \leqslant c. \max\{|f_0(z)|, \dots, |f_k(z)|\} \text{ for any } s = (k+1), \dots, n.$$

Hence

$$\|f(z)\| = \max\{|f_0(z)|, \dots, |f_n(z)|\} \leqslant c \max\{|f_0(z)|, \dots, |f_k(z)|\} = c\|f^*(z)\|,$$

where c is a positive constant, depends only on the $b_{s,i}$ and not on z and f^*. This implies

$$\begin{aligned}T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log \|f^*(re^{i\theta})\| d\theta + O(1)\\ = T_{f^*}(r) + O(1).\end{aligned}$$

Obviously $T_{f^*}(r) \leqslant T_f(r)$, so we have

$$T_{f^*}(r) = T_f(r) + O(1). \tag{3.2}$$

For any $z \in \mathbb{C}$, let $d_0(z), \dots, d_k(z)$ are the characteristic exponents of $f_0, \dots, f_n$ at z, of course it is also a characteristic exponents of $f_0, \dots, f_k$ at z_0 by definition. For any $j \in \{1, \dots, q\}$, by the construction of f^* and linear form L_j^*, we have

$$L_j \circ f(z) = L_j^* \circ f^*(z).$$

So $\mathrm{ord}_{L_j(f)}(z) = \mathrm{ord}_{L_j^*(f^*)}(z)$, this implies that

$$\nu(H_j, z) = \nu(H_j^*, z) \tag{3.3}$$

$$\varepsilon(H_j, z) = \varepsilon(H_j^*, z). \tag{3.4}$$

Since (3.3) we have

$$\mathcal{N}_f(r, H_j) = \mathcal{N}_{f^*}(r, H_j^*) \tag{3.5}$$

for any $j = 1, \dots, q$. Furthemore, since $W_f = W(f_0, \dots, f_k)$ so from (3.4) we have $\mathcal{V}(\mathcal{H}, z) = \mathcal{V}(\mathcal{H}^*, z)$ for any $z \in \mathbb{C}$. This implies that

$$\mathcal{U}_f(r, H_j) = \mathcal{U}_{f^*}(r, H_j^*) \tag{3.6}$$

Combining (3.1), (3.2), (3.5), (3.6) we have the conclusion of the theorem.

4. Conclusion

In this paper, we have stated and proved a new result about second main theorem for holomorphic curves from $\mathbb{C}$ into a linear projective subspace for the reduced counting function intersecting hyperplanes in general position with respect to subspace. Obviously reduced counting functions is less than truncated counting functions by k which is dimension of subspace, so we can replace the reduced counting functions by truncated counting functions by k on the right. Hence our theorem can be used to prove of the unique problem for holomorphic curves.

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