A SELF-ADAPTIVE ITERATIVE ALGORITHM FOR SOLVING THE SPLIT VARIATIONAL INEQUALITY PROBLEM IN HILBERT SPACES

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ABSTRACT

The split variational inequality problem (SVIP) was first introduced by Censor et al. Up to now, there is a long list of works concerning algorithms to solve (SVIP). In this paper, we study the split variational inequality problem in Hilbert spaces. In order to solve this problem, we propose a self-adaptive algorithm. Our algorithm uses dynamic step-sizes, chosen based on information of the previous step and their strong convergence is proved. In comparison with the work by Censor et al. (Numer. Algor., **59**:301–323, 2012), the new algorithm gives strong convergence results and does not require information about the spectral radius of the operator. And then, we give a numerical experiment to illustrate the performance of our algorithm.

THUẬT TOÁN LẶP TỰ THÍCH NGHI GIẢI BÀI TOÁN BẤT ĐẲNG THỨC BIẾN PHÂN TÁCH TRONG KHÔNG GIAN HILBERT

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Bài toán chấp nhận tách
Bất đẳng thức biến phân
Không gian Hilbert
Ánh xạ không giãn
Điểm bất động

TÓM TẮT

Bài toán bất đẳng thức biến phân tách (SVIP) được nghiên cứu đầu tiên bởi Censor và các cộng sự. Đến nay, có rất nhiều công trình nghiên cứu các thuật toán để giải bài toán SVIP. Trong bài báo này, chúng tôi đề cập đến bài toán bất đẳng thức biến phân tách trong không gian Hilbert. Để giải bài toán, chúng tôi trình bày một thuật toán tự thích nghi, sử dụng cỡ bước được chọn dựa trên thông tin của các bước lặp trước đó, đồng thời chứng minh sự hội tụ mạnh của thuật toán. So với công trình nghiên cứu của tác giả Censor (Numer. Algor., **59**:301–323, 2012), thuật toán mới của chúng tôi cho kết quả hội tụ mạnh và không cần sử dụng bán kính phổ của toán tử. Cuối cùng, chúng tôi đưa ra ví dụ minh họa cho phương pháp đã đề xuất.

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1. Introduction

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces with inner product $\langle .,. \rangle$ and norm $\|.\|$. *Variational Inequality Problem* (VIP) [1], [2] is the problem of finding a point u^* in a subset C of a Hilbert space $\mathscr{H}$ such that

$$\langle Au^*, u-u^* \rangle \geq 0 \quad \forall u \in C, \tag{VIP(A,C)}$$

where $A : C \to \mathscr{H}$ is a mapping, and we denote solution set of (VIP(A,C)) by $S_{(A,C)}$.

The *Split Feasibility Problem* (SFP) proposed by Censor and Elfving [3] is finding a point

$$u^* \in C \quad \text{and} \quad Fu^* \in Q, \tag{SFP}$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively and $F : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator.

In this paper we discuss a self-adaptive algorithm for solving the *Split Variational Inequality Problem* which was studied by Censor et al. in [4]

$$\text{find } u^* \in S_{(A,C)} \quad \text{and} \quad Fu^* \in S_{(B,Q)}. \tag{SVIP}$$

To solve the (SVIP), Censor et al. [4] presented a weak convergence result when A and B are η_A, η_B-inverse strongly monotone operators on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively.

In the present article, our aim is to introduce an iterative algorithm to solve the (SVIP) by using the viscosity approximation method [5], cyclic iterative methods [3], [6], [7] and a modification of the CQ–algorithm [8], [9]. We prove the strong convergence of the presented algorithm under some mild conditions. Particularly, in our method, the step size is selected in such a way that its implementation does not need any prior information on the norm of the transfer operators.

2. Preliminaries

In this section, we introduce some mathematical symbols, definitions, and lemmas which can be used in the proof of our main result.

Let $\mathscr{H}$ be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$ and C be a nonempty, closed and convex subset of $\mathscr{H}$. In what follows, we write $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}$ converges weakly to x while $x^k \to x$ indicate that the sequence $\{x^k\}$ converges strongly to x. It is known that in a Hilbert space $\mathscr{H}$,

$$2\langle x,y \rangle = \|x+y\|^2 - \|x\|^2 - \|y\|^2 = \|x\|^2 + \|y\|^2 - \|x-y\|^2, \tag{1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{2}$$

for all $x, y \in \mathscr{H}$ and $\lambda \in \mathbb{R}$ (see, for example [10, Lemma 2.13], [11]). For each $x \in \mathscr{H}$ there exists a mapping $P_C : \mathscr{H} \to C$ such that $\|x - P_C x\| \leq \|x-y\|$ $\forall x, y \in C$. The mapping P_C is called the metric projection of $\mathscr{H}$ onto C.

Lemma 2.1. (see [12]) *(i) P_C is a nonexpansive mapping.*

(ii) $P_C x \in C \quad \forall x \in \mathscr{H}$ and $P_C x = x \quad \forall x \in C$.

(iii) $x \in \mathscr{H}$, $y = P_C x$ *if and only if* $y \in C$ *and* $\langle x-y, z-y\rangle \le 0 \quad \forall z \in C$.

Definition 2.1. An operator $T : \mathscr{H} \to \mathscr{H}$ is called a contraction operator with the contraction coefficient $\tau \in [0,1)$ if $\|Tx - Ty\| \le \tau\|x-y\| \quad \forall x,y \in \mathscr{H}$.

It is easy to see that, if T is a contraction operator, then $P_C T$ is a contraction operator too. If $\tau \ge 0$ we have τ-Lipschitz continuous operator.

Definition 2.2. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces and let $F : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator. An operator $F^* : \mathscr{H}_2 \to \mathscr{H}_1$ with the property $\langle Fx, y\rangle = \langle x, F^*y\rangle$ for all $x \in \mathscr{H}_1$ and $y \in \mathscr{H}_2$, is called an adjoint operator of F.

The adjoint operator of a bounded linear operator F on a Hilbert space always exists and is uniquely determined. Furthermore, F^* is a bounded linear operator.

Definition 2.3. An operator $A : \mathscr{H} \to \mathscr{H}$ is called an η-inverse strongly monotone operator with constant $\eta > 0$ if $\langle Ax - Ay, x-y\rangle \ge \eta\|Ax - Ay\|^2 \quad \forall x,y \in \mathscr{H}$.

It is easy to see that, if A is an η-inverse strongly monotone operator, then $I^{\mathscr{H}} - \lambda A$ is a nonexpansive mapping for $\lambda \in (0, 2\eta]$.

Lemma 2.2. (see [4]) *Let $A : C \to \mathscr{H}$ be η-inverse strongly monotone on C and $\lambda > 0$ be a constant satisfying $0 < \lambda \le 2\eta$. Define the mapping $T : C \to C$ by taking*

$$Tx = P_C\big(I^{\mathscr{H}} - \lambda A\big)x \quad \forall x \in C. \tag{3}$$

Then T is nonexpansive mapping on C, furthermore, $\mathrm{Fix}(T) = S_{(A,C)}$ is the set of fixed points of T, where $\mathrm{Fix}(T) := \{x \in C \mid Tx = x\}$.

Lemma 2.3. (see [12]) *Assume that T be a nonexpansive mapping of a closed and convex subset C of a Hilbert space $\mathscr{H}$ into $\mathscr{H}$. Then the mapping $I^{\mathscr{H}} - T$ is demiclosed on C; that is, whenever $\{x^k\}$ is a sequence in C which weakly converges to some point $u^* \in C$ and the sequence $\{(I^{\mathscr{H}} - T)x^k\}$ strongly converges to some y, it follows that $(I^{\mathscr{H}} - T)u^* = y$.*

From Lemma , if $x^k \rightharpoonup u^*$ and $(I^{\mathscr{H}} - T)x^k \to 0$, then $u^* \in \mathrm{Fix}(T)$.

Lemma 2.4. (See [2]) *Let $\{s_k\}$ be a real sequence which does not decrease at infinity in the sense that there exists a subsequence $\{s_{k_n}\}$ such that $s_{k_n} \le s_{k_n+1}$ $\forall n \ge 0$. Define an integer sequence by $\nu(k) := \max\big\{k_0 \le n \le k \mid s_n < s_{n+1}\big\}$, $k \ge k_0$. Then $\nu(k) \to \infty$ as $k \to \infty$ and for all $k \ge k_0$, we have $\max\{s_{\nu(k)}, s_k\} \le s_{\nu(k)+1}$.*

Lemma 2.5. (see [13]) *Let $\{s_k\}$ be a sequence of nonnegative numbers satisfying the condition $s_{k+1} \le (1-b_k)s_k + b_k c_k$, $k \ge 0$, where $\{b_k\}$ and $\{c_k\}$ are sequences of real numbers such that*

(i) $\{b_k\} \subset (0,1)$ *for all* $k \ge 0$ *and* $\sum_{k=1}^{\infty} b_k = \infty$,

(ii) $\limsup_{k\to\infty} c_k \le 0$.

Then, $\lim_{k\to\infty} s_k = 0$.

3. Main Results

In this section, we use the viscosity approximation method and a modification of the *CQ*–algorithm to establish the strong convergence of the proposed algorithm for finding the solution of the (SVIP). We consider the (SVIP) under the following conditions.

Assumption 3.1.

(A1) $A : \mathscr{H}_1 \to \mathscr{H}_1$ is η_A-inverse strongly monotone on $\mathscr{H}_1$.
(A2) $B : \mathscr{H}_2 \to \mathscr{H}_2$ is η_B-inverse strongly monotone on $\mathscr{H}_2$.
(A3) $F : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator.
(A4) $T : \mathscr{H}_1 \to \mathscr{H}_1$ is a contraction operator with the contraction coefficient $\tau \in [0,1)$.
(A5) The solution set Ω_{SVIP} of (SVIP) is not empty.

If A and B satisfy the properties **(A1)** and **(A2)**, respectively, the solution sets $S_{(A,C)}$ and $S_{(B,Q)}$ are closed and convex. Here, for the sake of convenience, an empty set is considered to be closed and convex. Therefore, the solution set Ω_{SVIP} of the (SVIP) is also closed and convex.

Algorithm 1

Step 0. Select the initial point $x^1 \in \mathscr{H}_1$ and the sequences $\{\alpha_k\}$, $\{\beta_k\}$, $\{\rho_k\}$, $\{\kappa_k\}$, and λ such that the conditions

$$\{\alpha_k\} \subset (0,1),\ \alpha_k \to 0 \text{ as } k \to \infty, \text{ and } \sum_{k=1}^{\infty} \alpha_k = \infty, \tag{C1}$$

$$0 < \lambda \leq 2\eta,\ \eta = \min\{\eta_A, \eta_B\}, \tag{C2}$$

$$\{\beta_k\} \subset [a,b] \subset (0,1), \tag{C3}$$

$$\{\rho_k\} \subset [c,d] \subset (0,1),\ \{\kappa_k\} \subset (0,K),\ K > 0 \tag{C4}$$

are satisfied. Set $k := 1$.
Step 1. Compute $y^k = \beta_k x^k + (1-\beta_k)P_C\big(I^{\mathscr{H}_1} - \lambda A\big)x^k$.
Step 2. Compute $z^k = P_Q\big(I^{\mathscr{H}_2} - \lambda B\big)Fy^k$.
Step 3. Compute $v^k = y^k + \gamma_k F^*(z^k - Fy^k)$, where the step size γ_k is defined by

$$\gamma_k = \rho_k \frac{\|z^k - Fy^k\|^2}{\|F^*(z^k - Fy^k)\|^2 + \kappa_k}. \tag{4}$$

Step 4. Compute $x^{k+1} = \alpha_k T x^k + (1-\alpha_k)v^k$.
Step 5. Set $k := k+1$ and go to **Step 1**.

Theorem 3.1. *Suppose that all conditions in Assumption 3.1 are satisfied. Then the sequence* $\{x^k\}$ *generated by Algorithm 1 converges strongly to the unique solution of the* $\text{VIP}(I^{\mathscr{H}_1} - T, \Omega_{\text{SVIP}})$.

Proof. Since T is a contraction mapping, $P_{\Omega_{\text{SVIP}}}T$ is a contraction too. By Banach contraction operator principle, there exists a unique point $u^* \in \Omega_{\text{SVIP}}$ such that $P_{\Omega_{\text{SVIP}}}Tu^* = u^*$. By Lemma 2.1$(iii)$, we obtain u^* is the unique solution to the $\text{VIP}(I^{\mathscr{H}_1} - T, \Omega_{\text{SVIP}})$. Since $u^* \in \Omega_{\text{SVIP}}$, $u^* \in S_{(A,C)}$ and $Fu^* \in S_{(B,Q)}$.

Let $u \in \Omega_{\text{SVIP}}$, $u \in S_{(A,B)}$. Since Lemma 2.2, $u = P_C\big(I^{\mathscr{H}_1} - \lambda A\big)u$. From Step 1 in Algorithm 1, the nonexpansive property of $P_C\big(I^{\mathscr{H}_1} - \lambda A\big)$ (see Lemma 2.2), and (2), we have that

$$\begin{aligned}
\|y^k - u\|^2 &= \Big\|\beta_k(x^k - u) + (1-\beta_k)\Big[P_C\big(I^{\mathscr{H}_1} - \lambda A\big)x^k - P_C\big(I^{\mathscr{H}_1} - \lambda A\big)u\Big]\Big\|^2 \\
&\leq \beta_k\|x^k - u\|^2 + (1-\beta_k)\|x^k - u\|^2 - \beta_k(1-\beta_k)\big\|x^k - P_C\big(I^{\mathscr{H}_1} - \lambda A\big)x^k\big\|^2 \\
&= \|x^k - u\|^2 - \beta_k(1-\beta_k)\big\|x^k - P_C\big(I^{\mathscr{H}_1} - \lambda A\big)x^k\big\|^2 \qquad (5) \\
&\leq \|x^k - u\|^2. \qquad (6)
\end{aligned}$$

It follows from Step 3 in Algorithm 1, the property of adjoint operator F^*, and (1) that

$$\begin{aligned}\|v^k-u\|^2 &= \left\|y^k+\gamma_kF^*\left(z^k-Fy^k\right)-u\right\|^2\\ &= \|y^k-u\|^2+\gamma_k^2\left\|F^*\left(z^k-Fy^k\right)\right\|^2+2\gamma_k\left\langle y^k-u,F^*\left(z^k-Fy^k\right)\right\rangle\\ &= \|y^k-u\|^2+\gamma_k^2\left\|F^*\left(z^k-Fy^k\right)\right\|^2+2\gamma_k\left\langle Fy^k-Fu,z^k-Fy^k\right\rangle\\ &= \|y^k-u\|^2+\gamma_k^2\|F^*\left(z^k-Fy^k\right)\|^2+\gamma_k\left(\left\|z^k-Fu\right\|^2-\left\|Fy^k-Fu\right\|^2-\left\|z^k-Fy^k\right\|^2\right).\end{aligned}$$

Since $u\in\Omega_{\text{SVIP}}$, $Fu\in S_{(B,Q)}$. It follows from Lemma 2.2 that $Fu=P_Q\left(I^{\mathscr{H}_2}-\lambda B\right)Fu$. From Steps 2 and 3 in Algorithm 1, the nonexpansive property of $P_Q\left(I^{\mathscr{H}_2}-\lambda B\right)$, (4), (C4), and the last inequality, we obtain

$$\begin{aligned}\|v^k-u\|^2 &= \|y^k-u\|^2+\gamma_k^2\left\|F^*\left(z^k-Fy^k\right)\right\|^2\\ &\quad+\gamma_k\left(\left\|P_Q\left(I^{\mathscr{H}_2}-\lambda B\right)Fy^k-P_Q\left(I^{\mathscr{H}_2}-\lambda B\right)Fu\right\|^2-\left\|Fy^k-Fu\right\|^2-\left\|z^k-Fy^k\right\|^2\right)\\ &\le \|y^k-u\|^2+\gamma_k^2\left\|F^*\left(z^k-Fy^k\right)\right\|^2+\gamma_k\left(\left\|Fy^k-Fu\right\|^2-\left\|Fy^k-Fu\right\|^2-\left\|z^k-Fy^k\right\|^2\right)\\ &= \|y^k-u\|^2+\gamma_k^2\left\|F^*\left(z^k-Fy^k\right)\right\|^2-\gamma_k\left\|z^k-Fy^k\right\|^2\\ &\le \|y^k-u\|^2+\rho_k^2\frac{\left\|z^k-Fy^k\right\|^4}{\left(\left\|F^*(z^k-Fy^k)\right\|^2+\kappa_k\right)^2}\left(\left\|F^*\left(z^k-Fy^k\right)\right\|^2+\kappa_k\right)-\rho_k\frac{\left\|z^k-Fy^k\right\|^4}{\left\|F^*(z^k-Fy^k)\right\|^2+\kappa_k}\\ &= \|y^k-u\|^2-\rho_k(1-\rho_k)\frac{\left\|z^k-Fy^k\right\|^4}{\left\|F^*(z^k-Fy^k)\right\|^2+\kappa_k} \qquad (7)\\ &\le \|y^k-u\|^2. \qquad (8)\end{aligned}$$

It follows from the convexity of the norm function $\|.\|$ on $\mathscr{H}_1$, the contraction property of T with the contraction coefficient $\tau\in[0,1)$, (6), (8), and Step 4 in Algorithm 1 that

$$\begin{aligned}\|x^{k+1}-u\| &= \left\|\alpha_k\left(Tx^k-u\right)+(1-\alpha_k)(v^k-u)\right\|\le\alpha_k\left(\left\|Tx^k-Tu\right\|+\left\|Tu-u\right\|\right)+(1-\alpha_k)\|v^k-u\|\\ &\le \tau\alpha_k\|x^k-u\|+\alpha_k\left\|Tu-u\right\|+(1-\alpha_k)\|x^k-u\|\\ &= \left[1-(1-\tau)\alpha_k\right]\|x^k-u\|+(1-\tau)\alpha_k\frac{\left\|Tu-u\right\|}{1-\tau}\\ &\le \max\left\{\|x^k-u\|,\frac{\left\|Tu-u\right\|}{1-\tau}\right\}\le\cdots\le\max\left\{\|x^0-u\|,\frac{\left\|Tu-u\right\|}{1-\tau}\right\}.\end{aligned}$$

This implies that the sequence $\{x^k\}$ is bounded. Since P_C and P_Q are nonexpansive mappings and F is the bounded linear operator, we also have the sequences $\{y^k\}$, $\{z^k\}$, and $\{v^k\}$ are bounded.

Now we claim that $\lim_{n\to\infty}\|x^k-u^*\|=0$, where u^* is the unique solution of the VIP$(I^{\mathscr{H}_1}-T,\Omega_{\text{SVIP}})$, that is, $u^*=P_{\Omega_{\text{SVIP}}}Tu^*$. Indeed, from the convexity of $\|.\|^2$, Step 4 in Algorithm 1, (5), (7) with u replaced by u^*, and the condition (C1), we get

$$\begin{aligned}\|x^{k+1}-u^*\|^2 &= \left\|\alpha_k(Tx^k-u^*)+(1-\alpha_k)(v^k-u^*)\right\|^2\le\alpha_k\left\|Tx^k-u^*\right\|^2+(1-\alpha_k)\|v^k-u^*\|^2\\ &\le \alpha_k\left\|Tx^k-u^*\right\|^2+\|y^k-u^*\|^2-\rho_k(1-\rho_k)\frac{\left\|z^k-Fy^k\right\|^4}{\left\|F^*(z^k-Fy^k)\right\|^2+\kappa_k}\\ &\le \alpha_k\left\|Tx^k-u^*\right\|^2+\|x^k-u^*\|^2-\rho_k(1-\rho_k)\frac{\left\|z^k-Fy^k\right\|^4}{\left\|F^*(z^k-Fy^k)\right\|^2+\kappa_k}\\ &\quad-\beta_k(1-\beta_k)\left\|x^k-P_C\left(I^{\mathscr{H}_1}-\lambda A\right)x^k\right\|^2.\end{aligned}$$

Hence,

$$\rho_k(1-\rho_k)\frac{\left\|z^k-Fy^k\right\|^4}{\left\|F^*(z^k-Fy^k)\right\|^2+a_k}+\beta_k(1-\beta_k)\left\|x^k-P_C\big(I^{\mathscr{H}_1}-\lambda A\big)x^k\right\|^2$$
$$\leq\left(\|x^k-u^*\|^2-\|x^{k+1}-u^*\|^2\right)+\alpha_k\left\|Tx^k-u^*\right\|^2. \tag{9}$$

Next, from Step 4 in Algorithm 1 and the contraction property of T with the contraction coefficient $\tau\in[0,1)$, we have that

$$\|x^{k+1}-u^*\|^2=\langle\alpha_k(Tx^k-u^*)+(1-\alpha_k)(v^k-u^*),x^{k+1}-u^*\rangle$$
$$=(1-\alpha_k)\langle v^k-u^*,x^{k+1}-u^*\rangle+\alpha_k\langle Tx^k-u^*,x^{k+1}-u^*\rangle$$
$$\leq\frac{1-\alpha_k}{2}\left(\|v^k-u^*\|^2+\|x^{k+1}-u^*\|^2\right)+\alpha_k\langle Tx^k-Tu^*,x^{k+1}-u^*\rangle+\alpha_k\langle Tu^*-u^*,x^{k+1}-u^*\rangle$$
$$\leq\frac{1-\alpha_k}{2}\left(\|v^k-u^*\|^2+\|x^{k+1}-u^*\|^2\right)+\frac{\alpha_k}{2}\left(\tau\|x^k-u^*\|^2+\|x^{k+1}-u^*\|^2\right)+\alpha_k\langle Tu^*-u^*,x^{k+1}-u^*\rangle.$$

This implies that $\|x^{k+1}-u^*\|^2\leq(1-\alpha_k)\|v^k-u^*\|^2+\alpha_k\tau\|x^k-u^*\|^2+2\alpha_k\langle Tu^*-u^*,x^{k+1}-u^*\rangle$. From (6), (8) with u replaced by u^*, and the last inequality, we obtain

$$\|x^{k+1}-u^*\|^2\leq\big[1-(1-\tau)\alpha_k\big]\|x^k-u^*\|^2+2\alpha_k\langle Tu^*-u^*,x^{k+1}-u^*\rangle. \tag{10}$$

We consider two cases.

Case 1. There exists an integer $k_0\geq 0$ such that $\|x^{k+1}-u^*\|\leq\|x^k-u^*\|$ for all $k\geq k_0$.

Then, $\lim_{k\to\infty}\|x^k-u^*\|$ exists. From the boundedness of the sequence $\{Tx^k\}$, the conditions (C1), (C3), and (C4), it follows from (9) that

$$\lim_{k\to\infty}\left\|x^k-P_C\big(I^{\mathscr{H}_1}-\lambda A\big)x^k\right\|=0, \tag{11}$$

and

$$\lim_{k\to\infty}\|z^k-Fy^k\|=0. \tag{12}$$

From Step 1 in Algorithm 1 and (C3), we get

$$\lim_{k\to\infty}\|x^k-y^k\|=(1-\beta_k)\lim_{k\to\infty}\left\|x^k-P_C\big(I^{\mathscr{H}_1}-\lambda A\big)x^k\right\|=0. \tag{13}$$

Hence,

$$\lim_{k\to\infty}\left\|\big[I^{\mathscr{H}_1}-P_C\big(I^{\mathscr{H}_1}-\lambda A\big)\big]x^k\right\|=0. \tag{14}$$

From Step 2 in Algorithm 1 and (12), we obtain

$$\lim_{k\to\infty}\left\|\big[I^{\mathscr{H}_2}-P_Q\big(I^{\mathscr{H}_2}-\lambda B\big)\big]Fy^k\right\|=0. \tag{15}$$

From Step 3 in Algorithm 1, the property of adjoint operator F^*, and (12), we obtain

$$\|v^k-y^k\|=\gamma_k\|F^*(z^k-Fy^k)\|\to 0\quad\text{as}\quad k\to\infty. \tag{16}$$

It follows from (13) and (16) that

$$\|x^k-v^k\|\to 0\quad\text{as}\quad k\to\infty. \tag{17}$$

Using the boundedness of $\{v^k\}$ and $\{Tx^k\}$, Step 4 in Algorithm 1, and the condition (C1), we also have $\|x^{k+1}-v^k\| = \alpha_k\|Tx^k - v^k\| \to 0 \quad \text{as} \quad k\to\infty$. When combined with (17), this implies that

$$\|x^{k+1}-x^k\| \to 0 \quad \text{as} \quad k\to\infty. \tag{18}$$

Now we show that $\limsup_{k\to\infty}\langle Tu^*-u^*, x^{k+1}-u^*\rangle \le 0$. Indeed, suppose that $\{x^{k_n}\}$ is a subsequence of $\{x^k\}$ such that

$$\limsup_{k\to\infty}\langle Tu^*-u^*, x^k-u^*\rangle = \lim_{k_n\to\infty}\langle Tu^*-u^*, x^{k_n}-u^*\rangle. \tag{19}$$

Since $\{x^{k_n}\}$ is bounded, there exists a subsequence $\{x^{k_{n_l}}\}$ of $\{x^{k_n}\}$ which converges weakly to some point $u^\dagger$. Without loss of generality, we may assume that $x^{k_n} \rightharpoonup u^\dagger$. We will prove that $u^\dagger \in \Omega_{\text{SVIP}}$. Indeed, from (14), Lemmas 2.2 and 2.3, we obtain $u^\dagger \in S_{(A,C)}$. Moreover, since F is a bounded linear operator, $Fx^{k_n} \rightharpoonup Fu^\dagger$. Using (17), Lemmas 2.2 and 2.3, we also obtain $Fu^\dagger \in S_{(B,Q)}$. Hence, $u^\dagger \in \Omega_{\text{SVIP}}$. So, from $u^* = P_{\Omega_{\text{SVIP}}}Tu^*$, (19), and Lemma 2.1(*iii*) we deduce that

$$\limsup_{k\to\infty}\langle Tu^*-u^*, x^k-u^*\rangle = \langle Tu^*-u^*, u^\dagger-u^*\rangle \le 0,$$

which combined with (18) gives

$$\limsup_{k\to\infty}\langle Tu^*-u^*, x^{k+1}-u^*\rangle \le 0. \tag{20}$$

Now, the inequality (10) can be rewritten in the form $\|x^{k+1}-u^*\|^2 \le (1-b_k)\|x^k-u^*\|^2 + b_kc_k$, $k\ge 0$, where $b_k = (1-\tau)\alpha_k$ and $c_k = \frac{2}{1-\tau}\langle Tu^*-u^*, x^{k+1}-u^*\rangle$. Since the condition (C1) and $\tau\in[0,1)$, $\{b_k\}\subset(0,1)$ and $\sum_{k=1}^\infty b_k = \infty$. Consequently, from $\tau\in[0,1)$ and (20), we have that $\limsup_{k\to\infty} c_k \le 0$. Finally, by Lemma 2.5, $\lim_{k\to\infty}\|x^k-u^*\| = 0$.

Case 2. There exists a subsequence $\{k_n\}$ of $\{k\}$ such that $\|x^{k_n}-u^*\| \le \|x^{k_n+1}-u^*\|$ for all $n\ge 0$.

Hence, by Lemma 2.4, there exists an integer, nondecreasing sequence $\{\nu(k)\}$ for $k\ge k_0$ (for some k_0 large enough) such that $\nu(k)\to\infty$ as $k\to\infty$,

$$\|x^{\nu(k)}-u^*\| \le \|x^{\nu(k)+1}-u^*\| \quad \text{and} \quad \|x^k-u^*\| \le \|x^{\nu(k)+1}-u^*\| \tag{21}$$

for each $k\ge 0$. From (10) with k replaced by $\nu(k)$, we have

$$0 < \|x^{\nu(k)+1}-u^*\|^2 - \|x^{\nu(k)}-u^*\|^2 \le 2\alpha_{\nu(k)}\langle Tu^*-u^*, x^{\nu(k)+1}-u^*\rangle.$$

Since $\alpha_{\nu(k)}\to 0$ and the boundedness of $\{x^{\nu(k)}\}$, we conclude that

$$\lim_{k\to\infty}\left(\|x^{\nu(k)+1}-u^*\|^2 - \|x^{\nu(k)}-u^*\|^2\right) = 0. \tag{22}$$

By a similar argument to Case 1, we obtain

$$\lim_{k\to\infty}\left\|\left[I^{\mathcal{H}_1} - P_C\left(I^{\mathcal{H}_1}-\lambda A\right)\right]x^{\nu(k)}\right\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\left\|\left[I^{\mathcal{H}_2} - P_Q\left(I^{\mathcal{H}_2}-\lambda B\right)\right]Fy^{\nu(k)}\right\| = 0.$$

Also we get $\|x^{\nu(k)+1}-u^*\|^2 \le \left[1-(1-\tau)\alpha_{\nu(k)}\right]\|x^{\nu(k)}-u^*\|^2 + 2\alpha_{\nu(k)}\langle Tu^*-u^*, x^{\nu(k)+1}-u^*\rangle$, where $\limsup_{n\to\infty}\langle Tu^*-u^*, x^{\nu(k)+1}-u^*\rangle \le 0$. Since the first inequality in (21) and $\alpha_{\nu(k)} > 0$, we have that $(1-\tau)\|x^{\nu(k)}-u^*\|^2 \le 2\langle Tu^*-u^*, x^{\nu(k)+1}-u^*\rangle$.

Thus, from $\limsup_{n\to\infty}\langle Tu^*-u^*, x^{\nu(k)+1}-u^*\rangle \le 0$ and $\tau\in[0,1)$, we get $\lim_{k\to\infty}\|x^{\nu(k)}-u^*\|^2 = 0$. This together with (22) implies that $\lim_{k\to\infty}\|x^{\nu(k)+1}-u^*\|^2 = 0$. Which together with the second inequality in (21) implies that $\lim_{k\to\infty}\|x^k-u^*\| = 0$. This completes the proof. □

4. Numerical Results

We give a numerical experiment to illustrate the performance of our algorithm. This result is performed in Python running on a laptop Dell Latitude 7480 Intel core i5, 2.40 GHz 8GB RAM.

Example 4.1. Let $\mathscr{H}_1 = \mathbb{R}^3$ and $\mathscr{H}_2 = \mathbb{R}^4$. Operators $A : \mathbb{R}^3 \to \mathbb{R}^3$ and $B : \mathbb{R}^4 \to \mathbb{R}^4$ are defined by

$$Ax = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \quad \text{and} \quad Bx = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4$$

that are inverse strongly monotone operator with constant $\eta_A = \frac{1}{7}$ and $\eta_B = \frac{1}{3+\sqrt{3}}$. Bounded linear operator $F : \mathbb{R}^3 \to \mathbb{R}^4$, $Fx = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. And $Tx : \mathbb{R}^3 \to \mathbb{R}^3$, $Tx = \frac{1}{2}x$ is contractive operator with constant $\tau = \frac{1}{2}$. Let C and Q are defined by

$$C = \{x \in \mathbb{R}^3, \langle a_1, x \rangle \le b_1\}, \text{ with } a_1 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^\top, \; b_1 = 2;$$
$$Q = \{x \in \mathbb{R}^4, \langle a_2, x \rangle \le b_2\}, \text{ with } a_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^\top, \; b_2 = 3.$$

$\Omega_{\text{SVIP}} = \left\{ x = \begin{bmatrix} t & -t & 0 \end{bmatrix}^\top \mid t \in \mathbb{R} : t \ge -2 \right\}$. The unique solution of VIP$\left(I^{\mathbb{R}^3} - T, \Omega_{\text{SVIP}}\right)$ is $x^* = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$. Now, choose $\alpha_k = k^{-0.5}$, $\lambda = 0.25$, $\beta_k = 0.5$, $\rho_k = 0.25$ and $\kappa_k = 0.1$, tolerance $\varepsilon = 10^{-3}$ and initial point $x^1 = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}^\top$, we get

$$x = \begin{bmatrix} -6.78489854 \times 10^{-4} & 6.78489983 \times 10^{-4} & 2.71210451 \times 10^{-10} \end{bmatrix}^\top.$$

This result archived within 11.9×10^{-3} seconds.

Next, we used different choices of parameters. Table 1 shown below is the performance with different α_k parameter, $\lambda = 0.25$, $\beta_k = 0.5$, $\rho_k = 0.25$ and $\kappa_k = 0.1$.

Table1: Result with different α_k

ε	$\alpha_k = k^{-0.5}$			$\alpha_k = k^{-0.8}$		
	$\|x - x^*\|$	time (s)	k	$\|x - x^*\|$	time (s)	k
10^{-3}	0.96×10^{-3}	11.9×10^{-3}	53	0.99×10^{-3}	63.8×10^{-3}	632
10^{-6}	0.99×10^{-6}	33.9×10^{-3}	196	0.99×10^{-6}	857.7×10^{-3}	10688
10^{-9}	0.99×10^{-9}	54.8×10^{-3}	433	0.99×10^{-9}	7107.3×10^{-3}	64382

Then we changed the initial point, with the same choice of parameters, as $\alpha_k = k^{-0.5}$, $\lambda = 0.25$, $\beta_k = 0.5$, $\rho_k = 0.25$ and $\kappa_k = 0.1$. The results are recorded in Table 2.

Table2: Result with different initial vector

ε	$x^1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$			$x^1 = \begin{bmatrix} 9 & 9 & 9 \end{bmatrix}^\top$		
	$\|x - x^*\|$	time (s)	k	$\|x - x^*\|$	time (s)	k
10^{-3}	0.78×10^{-3}	2.9×10^{-3}	7	0.91×10^{-3}	3.9×10^{-3}	11
10^{-6}	0.93×10^{-6}	10.9×10^{-3}	51	0.99×10^{-6}	13.9×10^{-3}	98
10^{-9}	0.97×10^{-9}	34.9×10^{-3}	192	0.97×10^{-9}	41.8×10^{-3}	297

5. Conclusion

In this paper, we introduced a new algorithm (Algorithm 1) and a new strong convergence theorem (Theorem 3.1) for solving the (SVIP) in a real Hilbert spaces without prior knowledge of operators norms. We consider a numerical example to illustrate the effectiveness of the proposed algorithm.

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