Waldschmidt Constant of Certain sets of Points with 3 Supporting Lines in Projective Plane

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Abstract—The paper shows values of the initial degree and Waldschmidt constant for some special cases including several cases of ten points with three supporting lines in projective plane. These constants represent the complexity of optimal solutions in repeated path problems that have many applications in computer science, informatics theory and telecommunications.

Index Terms—Waldschmidt constant, initial degree, zero-dimensional scheme, fat points.

1. Introduction

We denote by $\mathbb{P}^n$ the projective space over an algebraically closed field k. Let $P \in \mathbb{P}^n$, we say that a form f of the polynomial ring $R := k[x_0, \ldots, x_n]$ has multiplicity at least m at P if all partial derivatives of f of order $< m$ vanishing at P.

Let $X := \{P_1, \ldots, P_s\} \subset \mathbb{P}^n$, let $m_1, \ldots, m_s$ be positive integers. Let $\mathcal{P}_i \subset R$ be the defining ideal of P_i consisting of all forms vanishing at P_i, for $1 \leq i \leq s$. We denote by $Z := m_1P_1 + \cdots + m_sP_s$ the zero-dimensional scheme corresponding to the ideal $J = \cap_{i=1}^{s} \mathcal{P}_i^{m_i}$ consisting of all forms of R vanishing at P_i with multiplicity at least m_i, for $i = 1, \ldots, s$. This zero-dimensional scheme is called a *fat point scheme*.

Let $A = \oplus_t A_t$, be any homogeneous ideal in $R := k[x_0, \ldots, x_n]$. The value $\alpha(A) = \min\{t | A_t \neq 0\}$ is called the initial degree of A. For the ideal $J = \cap_{i=1}^{s} \mathcal{P}^{m_i}$, the initial degree $\alpha(J)$ is the least degree of the hypersurfaces containing P_i with multiplicity at least m_i, for $1 \leq i \leq r$.

Definition. Let $X = \{P_1, \ldots, P_s\} \subset \mathbb{P}^n$, and $I = \cap_{i=1}^{s} \mathcal{P}_i \subset R = k[x_0, \ldots, x_n]$. For $m \in \mathbb{N}$, denote $I^{(m)} = \cap_{i=1}^{r} \mathcal{P}_i^m$, the ideal of the fat point scheme $Z = \sum_{i=1}^{s} mP_i$ with equal multiplicity m. The value

$$\lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}$$

is called the Waldschmidt constant of I or of the set X and denoted by $\gamma(I)$ or $\gamma(X)$.

It is not hard to see the following basic properties.

Lemma 1.1. *With notations as above, then*

1) $\alpha(I^{(m)}) \leq m\,\alpha(I)$.
2) $\gamma(I)$ *is well defined and* $1 \leq \gamma(I) \leq \dfrac{\alpha(I^{(m)})}{m} \leq \alpha(I), \forall m \geq 1$.
3) $\gamma(I) \leq \sqrt[n]{s}$.

Proof. See [3] and [18]. □

The constant is firstly introduced by Waldschmidt [21], [22]. Since then, many results of this constant was achieved mostly about finding lower bounds, see [2], [4]–[9], [11], [12], [16], [17]. That is recently one of active, fascinating and important topics as many applications in various areas of mathematics and other sciences, see [18] for more information. These constants represent the complexity of optimal solutions in repeated path problems that have many applications in computer science, informatics theory and telecommunications.

However, computation of $\alpha(I^{(m)})$ and $\gamma(I)$ is very hard in general, even for cases of small numbers of points in the projective plane. For a set of small number of points in general position in $\mathbb{P}^2$, the Waldschmidt constant was known only for cases of s points where $1 \leq s \leq 9$ or s is a perfect square, see [14], [15]. The value $\gamma(I)$ corresponding to the case of 10 generic points in $\mathbb{P}^2$ is still open. Recently, the constants were computed for certain sets of $r + s$ points where there are r collinear points and $1 \leq s \leq 7$, see [20]. Note that, if $s = 1$, it is the case of almost collinear as in [13]. The constants are also computed for certain sets of small

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number of points with two supporting lines in [19]. In this paper, we will compute the constants for some special cases including several cases of ten points with three supporting lines. We used many tools in computer algebra systems to support the computations.

For proofs in next section, we need to use following results of Bezout.

Theorem 1.2 ([10], I.7.7). *Let Y be a variety of dimension at least 1 in $\mathbb{P}^n$, and let H be a hypersurface not containing Y. Let $Z_1, \ldots, Z_s$ be the irreducible components of $Y \cap H$. Then*

$$\sum_{j=1}^{s} i(H, Y; Z_j) \deg Z_j = (\deg Y)(\deg H).$$

Note that $i(H, Y; Z_j)$ is the intersection multiplicity if Y and H along Z_j.

Corollary 1.3 (Bezout's Theorem, [10], I.7.8). *Let Y, Z be distinct curves in $\mathbb{P}^2$ having degrees d, e. Let $Y \cap Z = \{P_1, \ldots, P_s\}$. Then*

$$\sum_{j=1}^{s} i(H, Z; P_j) = de.$$

Note that $i(H, Z; P) \geq \mathrm{mult}_P(H) \cdot \mathrm{mult}_P(Z)$ and the equality holds if and only if H and Z have no tangent in common at P, see [3].

2. Main results

Let $X = \{P_1, \ldots, P_r\}$ be a set of points in $\mathbb{P}^2$; let l_1, l_2, l_3 be three lines containing all points of X, each line contains at least 3 points of X. The paper shows the constants of X in the following cases:

- no point of X lies on more than one of the lines;
- $r = 10$, one point of X lies in common of the three lines and each line contains exactly 4 points of X;
- $r = 10$, one point of X lies in common of two of the three lines and each line contains exactly 3 of 9 remain points;
- $r = 10$, one point of X is in common of the three lines and one of the lines contains 6 points of X.

Theorem 2.1. *Let $X = \{P_1, \ldots, P_r\}$, such that all the points lie on 3 lines l_1, l_2, l_3, each line contains at least 3 points of X and no point lies on more than one of $\{l_1, l_2, l_3\}$. Then $\alpha(I^{(m)}) = 3m$ for all $m \geq 1$ and $\gamma(X) = 3$. Consequently, that holds for ten points in the configuration.*

Proof. Let $I = \cap_{i=1}^{r} \mathcal{P}_i$. Let $l_1 = V(x), l_2 = V(z), l_3 = V(f_0)$ be the 3 lines, where f_0 is a linear form. It is easy to see that $x^m z^m f_0^m \in I^{(m)}$, thus $\alpha(I^{(m)}) \leq 3m$ for all $m \geq 1$. Then $\gamma(X) \leq 3$.

Prove that $I^{(m)}_{3m-1} = 0$ for any $m \geq 1$. For $m = 1$, it is clear that $I_2 = 0$. Let $f \in I^{(m)}_{3m-1}$. Since each line contains at least 3 points, it is easy to see that $x \mid f$, $z \mid f$ and $f_0 \mid f$ by Bezout's Theorem. We can write $f = xzf_0f_1$. Since no point of X lies on more than one line of $\{l_1, l_2, l_3\}$, we have $f_1 \in I^{(m-1)}_{3(m-1)-1}$. Then by induction, we have $I^{(m)}_{3m-1} = 0$. Thus $\alpha(I^{(m)}) \geq 3m$ and $\gamma(I) \geq 3$. Therefore we have desired equalities. □

Theorem 2.2. *Let $X = \{P_1, \ldots, P_{10}\}$, such that all the points lie on 3 lines l_1, l_2, l_3, each line contains exactly 4 points of X and $P_1 \in l_1 \cap l_2 \cap l_3$. Then $\alpha(I^{(m)}) = 3m$ for all $m \geq 1$ and $\gamma(X) = 3$.*

Proof. Let $I = \cap_{i=1}^{10} \mathcal{P}_i$, let $l_1 = V(x), l_2 = V(z), l_3 = V(f_0)$ be the 3 lines, where f_0 is a linear form, see Figure 1.

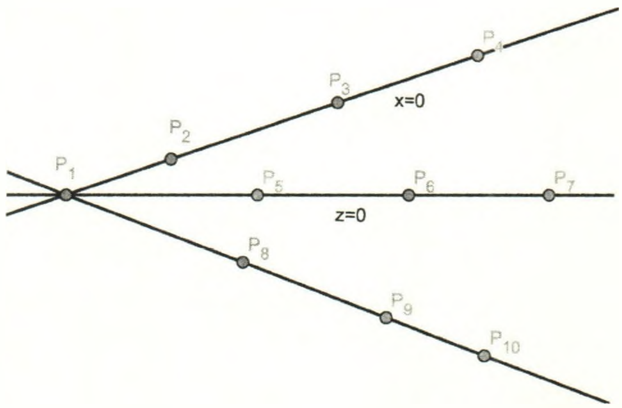

Fig. 1: Three lines with one point in common, 4 points on each line

It is easy to see that $x^m z^m f_0^m \in I^{(m)}$, thus $\alpha(I^{(m)}) \leq 3m$ for all $m \geq 1$ and $\gamma(I) \leq 3$.

We will show that $I^{(m)}_{3m-1} = 0$ for all $m \geq 1$. Let $f \in I^{(m)}_{3m-1}$, we see that $f \in J^{(m)}_{3m-1}$, where $J = \cap_{i=2}^{10} \mathcal{P}_i$. By Theorem 2.1, we have $J^{(m)}_{3m-1} = 0$. Thus $I^{(m)}_{3m-1} = 0$ for all $m \geq 1$. It means that $\alpha(I^{(m)}) \geq 3m$ for all $m \geq 1$. □

Corollary 2.3. *With the same notations and argument as above, the result holds when P_1 lies on only two of three lines as in Figure 2.*

Theorem 2.4. *Let $X = \{P_1, \ldots, P_{10}\}$. Suppose there are three distinct lines l_1, l_2, l_3 such that $\{P_1, P_2, P_3\} \subset l_1$; $\{P_1, P_9, P_{10}\} \subset l_2$ and $\{P_1, P_4, \ldots, P_8\} \subset l_3$. Then $\gamma(X) = 31/11$.*

Proof. Let $I = \cap_{i=1}^{10} \mathcal{P}_i$, let $l_1 = V(x), l_2 = V(f_0), l_3 = V(z)$ be the 3 lines, where f_0 is a linear form, see Figure 3.

Let C_i, for $4 \leq i \leq 8$ be the conic containing P_i and P_2, P_3, P_9, P_{10}. Let $C = V(g_0)$ be the irreducible curve of degree 11 containing P_1, having multiplicity at least 5 at P_2, P_3, P_9, P_{10} and multiplicity at least 2 at $P_4, \ldots, P_8$. (That curve exists by look at the number of coefficients). It is easy to see that $z^8 x f_0 C_4 C_5 C_6 C_7 C_8 g_0 \in I^{(11)}$, thus $\alpha(I^{(11)}) \leq 31$ and then $\gamma(I) \leq 31/11$.

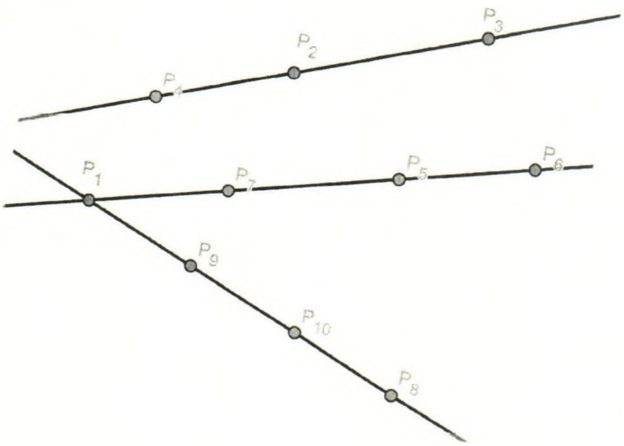

Fig. 2: Ten points in 3 line, two of lines contain one point in common

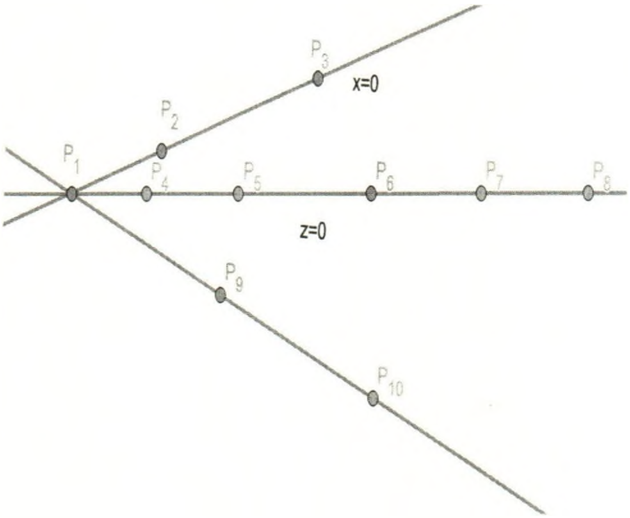

Fig. 3: Three lines with one point in common, one line with 6 points

We will show that $I^{(11m)}_{31m-1} = 0$ for all $m \geq 1$. Let $f \in I^{(11m)}_{31m-1}$. It is easy to see that $x, z, f_0 \mid f$. Then $f = z^a x^b f_0^c g$, where $x \nmid g$, $z \nmid g, f_0 \nmid g$. Let $\deg(g) = d$ then $a + b + c + d = 31m - 1$. We have $d \geq (11m - a - b - c) + 5(11m - a) = 66m - (a + b + c) - 5a = 66m - (31m - 1 - d) - 5a$ by Bezout's Theorem. This implies that $5a \geq 35m + 1$. We also have $d \geq 11m - a - b - c + 2(11m - b) = 33m - (31m - 1 - d) - 2b$ by Bezout's Theorem and obtain that $2b \geq 2m + 1$. Similarly, we have $2c \geq 2m + 1$.

Look at the conic C_i for $4 \leq i \leq 8$. If C_i is not a divisor of $V(g)$, we have $2d \geq 2(11m - b) + 2(11m - c) + (11m - a) = 55m - 2(b + c) - a$. This implies that $a \geq 55m - 2(31m - 1 - a) = -7m + 2 + 2a$ then $a \leq 7m - 2$. This contradicts to $5a \geq 35m + 1$. Therefore we have C_i is a divisor of $V(g)$.

Look at the curve $V(g_0)$ and suppose that g_0 is not a factor of g, then $11d \geq (11m - a - b - c) + 2\cdot 5(11m - b) + 2\cdot 5(11m - c) + 5\cdot 2(11m - a) = 31\cdot 11m - 11(a + b + c) = 31\cdot 11m - 11(31m - 1 - d) = 11d + 11$, impossible. Therefore we have g_0 is a factor of g. This means that $f = z^a x^b f_0^c C_4 \cdots C_8 g_0 f_1 \in I^{(11m)}_{31m-1}$. Note that, for all $m \geq 1$, we have $a \geq 8. b \geq 2, c \geq 2$. Then we can write $f = z^8 x f_0 C_4 \cdots C_8 g_0 h$, where $h \in I^{(11(m-1))}_{31(m-1)-1}$. Since $I^{(11m)}_{31m-1} = 0$ for $m = 1$, then by induction we have $I^{(11m)}_{31m-1} = 0$ for all $m \geq 1$. It means $\alpha(I^{(11m)}) \geq 31m$ for all $m \geq 1$ and then $\gamma(I) \geq 31/11$. Thus the equality follows. □

3. Conclusion

The paper shows computation of initial degree and Waldschmidt constant for some special cases in $\mathbb{P}^2$ including fours cases of ten points with three supporting lines. These results show the complexity of the optimal solutions in repeated path problems with the corresponding base point sets, which have many applications in computer science, informatics theory and telecommunications. Calculations for more complex configurations can be developed from these results.

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