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**Research Article**

# THE ARTINIANESS AND $(I, J)$-STABLE OF LOCAL HOMOLOGY MODULE WITH RESPECT TO A PAIR OF IDEALS

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**ABSTRACT**

*The concept of $I$-stable modules was defined by Tran Tuan Nam* (Tran, 2013)*, and the author used it to study the representation of local homology modules. In this paper, we will introduce the concept of $(I, J)$-stable modules, which is an extension of the $I$-stable modules. We study the $(I, J)$-stable for local homology modules with respect to a pair of ideals, these modules have been studied by Tran and Do (2020). We show some basic properties of $(I, J)$-stable modules and use them to study the artinianess of local homology modules with respect to a pair of ideals. Moreover, we also examine the relationship between the artinianess, $(I, J)$-stable, and the varnishing of local homology module with respect to a pair of ideals.*

***Keywords:*** artinian module; $I$-stable module; local homology

## 1. Introduction

Throughout this paper, $(R, \mathfrak{m})$ is a local noetherian ring with the maximal ideal $\mathfrak{m}$. Let $I, J$ be ideals of $R$. In (Tran & Do, 2020) we defined the local homology module $H_i^{I,J}(M)$ with respect to a pair of ideals $(I, J)$ by

$$H_i^{I,J}(M) = \varprojlim_{\mathfrak{a} \in W(I,J)} Tor_i^R(R/\mathfrak{a}, M)$$

in which $\tilde{W}(I, J)$ the set of ideals $\mathfrak{a}$ of $R$ such that $I^n \subseteq \mathfrak{a} + J$ for some integer $n$. This definition is dual to the generalized local cohomology as reported in a study by Takahashi, Yoshino, and Yoshizawa (2009) and an extension from the local homology module in a study by Nguyen and Tran (2001). We also studied some properties of these modules in a

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study by Tran and Do (2020), especially, we established the relationship between these modules and local homology modules with respect to an ideal through the isomorphic $H_i^{I,J}(M) \cong \varprojlim_{\mathfrak{a}\in\tilde{W}(I,J)} H_i^{\mathfrak{a}}(M)$. Tran (2013) introduced the definition of $I$-stable modules, and the author used it to study the representation of local homology modules.

In this paper, we will introduce the concept of $(I,J)$-stable module, which is an extension of the concept $I$-stable in Tran (2013)'s study. Also, we show some properties of artinian and $(I,J)$-stable of local homology modules $H_i^{I,J}(M)$. The first main result is Proposition 2.2, there is a $\mathfrak{b}\in\tilde{W}(I,J)$ such that $\mathfrak{b}\subseteq \bigcap_{\mathfrak{p}\in\mathrm{Coass}(M)} \mathfrak{p}$ where $M$ is $(I,J)$-separated artinian $R$-module. Next, Theorem 2.7 gives us the equivalent properties on artinianess of the local homology module. The last result gives the relationship between the artinianess, $(I,J)$-stable, and the varnishing of local homology module $H_i^{I,J}(M)$.

## 2. Some properties

***Lemma 2.1.*** *Let $M$ be an artinian $R$-module. Then $H_0^{I,J}(M)=0$ if and only if there is $x\in\mathfrak{b}$ such that $xM=M$ for some $\mathfrak{b}\in\tilde{W}(I,J)$.*

*Proof.* According to Tran and Do (2020), $H_0^{I,J}(M)\cong\Lambda_{I,J}(M)$ and by $M$ is artinian so there is $\mathfrak{b}\in\tilde{W}(I,J)$ such that $\Lambda_{I,J}(M)\cong M/\mathfrak{b}M$. Therefore, $H_0^{I,J}(M)=0$ if and only if $\mathfrak{b}M=M$ and by (Macdonald, 1973) if and only if $xM=M$ for $x\in\mathfrak{b}$.

We recall the concept of $(I,J)$-separated. The module $M$ is called $(I,J)$-separated if $\bigcap_{\mathfrak{a}\in\tilde{W}(I,J)} \mathfrak{a}M=0.$

***Proposition 2.2.*** *If $M$ is $(I,J)$-separated artinian $R$-module. Then there is a $\mathfrak{b}\in\tilde{W}(I,J)$ such that $\mathfrak{b}\subseteq \bigcap_{\mathfrak{p}\in\mathrm{Coass}(M)}\mathfrak{p}$.*

*Proof.* $M$ is $(I,J)$-separated, by (Tran & Do, 2020) $M\cong\Lambda_{I,J}(M)\cong M/\mathfrak{b}M$ for some $\mathfrak{b}\in\tilde{W}(I,J)$ hence $\mathfrak{b}M=0$. It implies that $\bigcap\mathfrak{b}^tM=0,$ so $M$ is $\mathfrak{b}$-separated. It follows Tran (2013) that $\mathfrak{b}\subseteq \bigcap_{\mathfrak{p}\in\mathrm{Coass}(M)}\mathfrak{p}$.

***Corollary 2.3.*** *Let $M$ is an artinian $R$-module. If $H_i^{I,J}(M)\neq 0,$ then there is a $\mathfrak{b}\in\tilde{W}(I,J)$ such that $\mathfrak{b}\subseteq \bigcap_{\mathfrak{p}\in\mathrm{Coass}(H_i^{I,J}(M))}\mathfrak{p}$.*

*Proof.* According to Tran and Do (2020), $H_i^{I,J}(M)$ is $(I,J)$-separated, and hence by Proposition 2.2, we have the conclusion.

***Corollary 2.4.*** *Let* $M$ *is an artinian* $R$*-module. If* $H_i^{I,J}(M)$ *is an artinian* $R$*-module, then there is an ideal* $\mathfrak{b} \in \tilde{W}(I,J)$ *such that* $\mathfrak{b} \subseteq \sqrt{\mathrm{Ann}_R H_i^{I,J}(M)}.$

*Proof.* According to Brodmann (1998), $\bigcap_{\mathfrak{p} \in \mathrm{Att}(H_i^{I,J}(M))} \mathfrak{p} = \sqrt{\mathrm{Ann}_R H_i^{I,J}(M)}.$ On the other hand, $H_i^{I,J}(M)$ is a representable, so $\mathrm{Att}(H_i^{I,J}(M)) = \mathrm{Coass}(H_i^{I,J}(M)),$ by (Yassemi, 1995). Now the conclusion follows from Corollary 2.3.

The concept of $I$-stable modules was defined in (Tran, 2009). An $R$-module $N$ is called $I$-stable if for each element $x \in I,$ there is a positive integer $n$ such that $x^t N = x^n N$ for all $t \geqslant n.$ Now we will give an extension concept of the $I$-stable.

***Definition 2.5.*** $M$ is called $(I,J)$-stable if there is an ideal $\mathfrak{b} \in \tilde{W}(I,J)$ such that $\bigcap_{\mathfrak{a} \in \tilde{W}(I,J)} \mathfrak{a}M = \mathfrak{b}M.$

When $J = 0$, we have $\mathfrak{b}M = \bigcap_{\mathfrak{a} \in \tilde{W}(I,J)} \mathfrak{a}M \subseteq \bigcap_{t>0} I^t M.$ Since $\mathfrak{b} \in \tilde{W}(I,J)$ and $J = 0$, it is $n$ such that $I^n \subseteq \mathfrak{b}.$ So $\bigcap_{t>0} I^t M = I^n M$, then $I^t M = I^n M$ for all $t > n.$ Hence, when $J = 0$ then $M$ is $I$-stable.

***Lemma 2.6.*** *Let* $0 \to M \to N \xrightarrow{g} P \to 0$ *be a short exact sequence in which the modules* $M, N, P$ *are* $(I,J)$*-separated. Then module* $N$ *is* $(I,J)$*-stable if and only if modules* $M, P$ *are* $(I,J)$*-stable.*

*Proof.* Assume that $N$ is $(I,J)$-stable. Then there is ideal $\mathfrak{b} \in \tilde{W}(I,J)$ such that $\mathfrak{b}N = \cap\, \mathfrak{a}N = 0,$ $N$ is $(I,J)$-saparated, $\mathfrak{b}M \subseteq \mathfrak{b}N = 0$, so $M$ is $(I,J)$-stable. We have $\mathfrak{b}P \cong (\mathfrak{b}N + \mathrm{Ker} g) / \mathrm{Ker} g = 0 = \cap \mathfrak{b}P,$ so $P$ is $(I,J)$-stable. Otherwise, suppose that $M$ and $P$ are $(I,J)$-stable, then there are ideals $\mathfrak{a}, \mathfrak{b}$ such that $\mathfrak{b}P = \mathfrak{a}M = 0.$ Let $d = \mathfrak{a} \cap \mathfrak{b}$, then $dM = dP = 0,$ (Brodmann, 1998), $d^n N = 0,$ so $N$ is $(I,J)$-stable.

***Proposition 2.7.*** *Let* $M$ *be an artinian* $R$*-module and* $t$ *a positive integer. Then the following statements are equivalent*

i) $H_i^{I,J}(M)$ *is an artinian for all* $i < t$;

ii) *There is an ideal* $\mathfrak{b} \in \tilde{W}(I,J)$ *such that* $\mathfrak{b} \subseteq \mathrm{Rad}(\mathrm{Ann}(H_i^{I,J}(M)))$ *for all* $i < t.$

*Proof.* $(i \Rightarrow ii)$ $H_i^{I,J}(M)$ is artinian, hence according to Tran and Do (2020), there is $\mathfrak{b} \in \tilde{W}(I,J)$ such that

$\mathfrak{b}H_i^{I,J}(M) = \bigcap_{\mathfrak{a} \in \tilde{W}(I,J)} \mathfrak{a}H_i^{I,J}(M) = 0.$ Therefore, $\mathfrak{b} \subseteq \mathrm{Rad}(\mathrm{Ann}(H_i^{I,J}(M)))$ for all $i < t$.

$(ii \Rightarrow i)$ We use induction on $t$. When $t = 1$, $H_0^{I,J}(M) \cong \Lambda_{I,J}(M) \cong M / \mathfrak{b}M$, so $H_0^{I,J}(M)$ is artinian. Let $t > 1$, according to Tran and Do (2020), we can replace $M$ by $\bigcap_{\mathfrak{a} \in \tilde{W}(I,J)} \mathfrak{a}M$. As $M$ is artinian, there is a $\mathfrak{b} \in \tilde{W}(I,J)$ such that $\mathfrak{b}M = \bigcap_{\mathfrak{a} \in \tilde{W}(I,J)} \mathfrak{a}M$. Therefore, we can assume that $M = \mathfrak{b}M$, according to MacDonal (1973), there is an element $x \in \mathfrak{b}$ such that $M = xM$. By the hypothesis, there is a positive integer $s$ such that $x^s H_i^{I,J}(M) = 0$ for all $i < t$. Then the short exact sequence

$$0 \longrightarrow (0 :_M x^s) \longrightarrow M \xrightarrow{x^s} M \longrightarrow 0$$

gives rise the exact sequence

$$0 \to H_{i+1}^{I,J}(M) \to H_i^{I,J}(0 :_M x^s) \to H_i^{I,J}(M) \to 0$$

for all $i < t - 1$. It follows a study by Brodmann (1998) that $\mathfrak{b} \subseteq \mathrm{Rad}(\mathrm{Ann}(H_i^{I,J}(0 :_M x^s)))$ and by the inductive hypothesis that $H_i^{I,J}(0 :_M x^s)$ is artinian for all $i < t - 1$. Thus $H_i^{I,J}(M)$ is artinian for all $i < t$.

We now recall the concept of the *Noetherian dimension* of an $R$-module $M$ denoted by $\mathrm{Ndim}\, M$. Note that the notion of the Noetherian dimension was introduced first by Roberts (1975) by the name Krull dimension. Later, Kirby (1990)changed this terminology of Roberts and referred to the Noetherian dimension to avoid confusion with the well-known Krull dimension of finitely generated modules. Let $M$ be an $R$-module. When $M = 0$, we put $\mathrm{Ndim}\, M = -1$. Then by induction, for any ordinal $\alpha$, we put $\mathrm{Ndim}\, M = \alpha$ when (i) $\mathrm{Ndim}\, M < \alpha$ is false, and (ii) for every ascending chain $M_0 \subseteq M_1 \subseteq \ldots$ of submodules of $M$, there exists a positive integer $m_0$ such that $\mathrm{Ndim}(M_{m+1} / M_m) < \alpha$ for all $m \geq m_0$. Thus $M$ is non-zero and finitely generated if and only if $\mathrm{Ndim}\, M = 0$.

***Theorem 2.8.*** *Let $M$ be an artinian $R$-module and $s$ an integer. Then the following statements are equivalent*

i) $H_i^{I,J}(M)$ *is* $(I,J)$*-stable for all* $i>s$;

ii) $H_i^{I,J}(M)$ *is artinian for all* $i>s$;

iii) $\mathrm{Ass}(H_i^{I,J}(M))\subseteq\{\mathfrak{m}\}$ *for all* $i>s$;

iv) $H_i^{I,J}(M)=0$ *for all* $i>s$.

*Proof.* $(i\Rightarrow ii)$ We use induction on $d=\mathrm{Ndim}M$. If $d=0, H_i^{I,J}(M)=0$ for all $i>0$, so $H_i^{I,J}(M)$ is artinian. Let $d>0$, we can replace $M$ by $\bigcap_{\mathfrak{a}\in\tilde{W}(I,J)}\mathfrak{a}M$ and $M$ is artinian; hence we may assume $M=\mathfrak{b}M$ for some $\mathfrak{b}\in\tilde{W}(I,J)$ and $\mathfrak{b}M$ is the minimum in the set $\{\mathfrak{a}M\mid\mathfrak{a}\in\tilde{W}(I,J)\}$. $H_i^{I,J}(M)$ is $(I,J)$-stable so there is an ideal $\mathfrak{c}\in\tilde{W}(I,J)$ such that $\mathfrak{c}H_i^{I,J}(M)=\bigcap_{\mathfrak{a}\in\tilde{W}(I,J)}\mathfrak{a}H_i^{I,J}(M)=0$. Let $d=\mathfrak{c}\cap\mathfrak{b}$, then $dM=\mathfrak{b}M=M$, hence there is $x\in d$ such that $xM=M$, and $xH_i^{I,J}(M)=0$. We have the short exact sequence $0\to(0:_M x)\to M\to M\to 0$ gives rise to the exact sequence

$$0\to H_{i+1}^{I,J}(M)\to H_i^{I,J}(0:_M x)\to H_i^{I,J}(M)\to 0.$$

Because $H_i^{I,J}(M)$ is $(I,J)$-stable for all $i>s$, so $H_i^{I,J}(0:_M x)$ is $(I,J)$-stable for all $i>s-1$. By the induction hypothesis $H_i^{I,J}(0:_M x)$ is artinian for all $i>s-1$. Therefore, $H_i^{I,J}(M)$ is artinian for all $i>s$.

$(ii\Rightarrow iii)$ (Yassemi, 1995), $\mathrm{Supp}(H_i^{I,J}(M))\supseteq\mathrm{Cosupp}(H_i^{I,J}(M))\cap\mathrm{Max}(R)=\{\mathfrak{m}\}$. Hence $\mathrm{Ass}(H_i^{I,J}(M))\subseteq\{\mathfrak{m}\}$.

$(iii\Rightarrow iv)$ We use induction on $d=\mathrm{Ndim}M$. When $d=0$, (Tran & Do, 2020), $H_i^{I,J}(M)=0$ for all $i>0$. Now, let $d>0$, we may assume that $M=xM$ for $x\in\mathfrak{b}$ and $\mathfrak{b}\in\tilde{W}(I,J)$. From the short exact sequence $0\to(0:_M x)\to M\to M\to 0$ gives rise to the exact sequence

$$H_{i+1}^{I,J}(M)\to H_i^{I,J}(0:_M x)\to H_i^{I,J}(M)\to H_i^{I,J}(M)$$

$\mathrm{Ass}(H_i^{I,J}(0:_M x))\subseteq\{\mathfrak{m}\}$ and $\mathrm{Ndim}(0:_M x)\leqslant d-1$. By the induction hypothesis $H_i^{I,J}(0:_M x)=0$ for all $i>s$. From that, we have the exact sequence

$0 \to H_i^{I,J}(M) \xrightarrow{.x} H_i^{I,J}(M)$. If $H_i^{I,J}(M) \neq 0,$ for all $i > s$, then $\mathrm{Ass}(H_i^{I,J}(M)) = \{\mathfrak{m}\}$, there is an element $a \in H_i^{I,J}(M)$ such that $\mathfrak{m} = \mathrm{Ann}(a)$ it implies that $a\mathfrak{m} = 0,$ so $xa = 0,$ hence $a = 0,$ it is a contraction. Therefore, $H_i^{I,J}(M) = 0$ for all $i > s.$

$(iv \Rightarrow i)$ It is clear.

## 3. Conclusion

In this paper, we gave the concept of the $(I,J)$*-stable module*. We studied the properties of the $(I,J)$*-stable* of local homology module with respect to a pair of ideals $(I,J).$ Moreover, we showed the relationship between of the artinianess and the $(I,J)$*-stable* of local homology module with respect to a pair of ideals.

❖ ***Conflict of Interest:*** *Authors have no conflict of interest to declare.*

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# TÍNH ARTIN VÀ TÍNH $(I, J)$-ỔN ĐỊNH CỦA MÔĐUN ĐỒNG ĐIỀU ĐỊA PHƯƠNG TƯƠNG ỨNG VỚI MỘT CẶP IĐÊAN

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**TÓM TẮT**

*Khái niệm về môđun $I$-ổn định được đưa ra bởi Tran Tuan Nam trong bài báo* (Tran, 2013) *và tác giả đã sử dụng nó như một công cụ để nghiên cứu tính biểu diễn được của lớp môđun đồng điều địa phương. Trong bài báo này, chúng tôi sẽ giới thiệu về lớp môđun $(I, J)$-ổn định, đây được xem như là một khái niệm mở rộng thực sự từ khái niệm $I$-ổn định. Chúng tôi nghiên cứu tính $(I, J)$-ổn định cho lớp môđun đồng điều địa phương theo một cặp iđêan, lớp môđun này đã được chúng tôi nghiên cứu trong* (Tran & Do, 2020)*. Các tính chất cơ bản về môđun $(I, J)$-ổn định đã được nghiên cứu và sử dụng nó để nghiên cứu tính artin của lớp môđun đồng điều địa phương theo một cặp iđêan. Hơn nữa, chúng tôi cũng đưa ra mối liên hệ giữa tính artin, tính $(I, J)$-ổn định và tính triệt tiêu của lớp môđun đồng điều địa phương theo một cặp iđêan.*

***Từ khóa:*** môđun artin; môđun $I$-ổn định; đồng điều địa phương