TRIGONOMETRIC-TYPE IDENTITIES AND THE PARITY OF BALANCING AND LUCAS-BALANCING NUMBERS

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ABSTRACT

Balancing numbers n are originally defined as the solution of the Diophantine equation $1+2+\cdots+(n-1)=(n+1)+\cdots+(n+r)$, where r is called the balancer corresponding to the balancing number n. By slightly modifying, n is the cobalancing number with the cobalancer r if $1+2+\cdots+n=(n+1)+\cdots+(n+r)$. Let B_n denote the n^{th} balancing number and b_n denote the n^{th} cobalancing number. Then $8B_n^2+1$ and $8b_n^2+8b_n+1$ are perfect squares. The n^{th} Lucas-balancing number C_n and the n^{th} Lucas-cobalancing number c_n are the positive roots of $8B_n^2+1$ and $8b_n^2+8b_n+1$, respectively. In this paper, we establish some trigonometric-type identities and some arithmetic properties concerning the parity of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers.

MỘT SỐ ĐẲNG THỨC KIỂU LƯỢNG GIÁC VÀ TÍNH CHẴN LẺ CỦA CÁC SỐ CÂN BẰNG VÀ CÁC SỐ LUCAS-CÂN BẰNG

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Đẳng thức kiểu lượng giác

Tính chẵn lẻ

TÓM TẮT

Các số cân bằng n được định nghĩa như là nghiệm của phương trình Diophantus $1+2+\cdots+(n-1)=(n+1)+\cdots+(n+r)$, trong đó r được gọi là hệ số cân bằng ứng với số cân bằng n. Tương tự như vậy, n là một số đối cân bằng với hệ số đối cân bằng r nếu $1+2+\cdots+n=(n+1)+\cdots+(n+r)$. Ký hiệu B_n là số cân bằng thứ n và b_n là số đối cân bằng thứ n. Khi đó, $8B_n^2+1$ và $8b_n^2+8b_n+1$ là những số chính phương. Số Lucas-cân bằng thứ n, ký hiệu C_n, và số Lucas-đối cân bằng thứ n, ký hiệu c_n, lần lượt là các căn bậc hai dương của $8B_n^2+1$ và $8b_n^2+8b_n+1$. Trong bài báo này, bằng những tính toán sơ cấp, chúng tôi thiết lập một số đẳng thức kiểu lượng giác và từ đó chỉ ra một số tính chất số học liên quan đến tính chẵn lẻ của các số cân bằng, các số đối cân bằng, các số Lucas-cân bằng và các số Lucas-đối cân bằng.

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1 Introduction

The study of number sequences has been a source of attraction to the mathematicians since ancient times. From that time many mathematicians have been focusing their attention on the study of the fascinating triangular numbers (numbers of the form $n(n+1)/2$ where $n \in \mathbb{Z}^+$ are known as triangular numbers) [1, 2, 3, 4]. While studying triangular numbers, Behera and Panda [4] introduced the notion of *balancing numbers*. An integer $n \in \mathbb{Z}^+$ is a balancing number if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r), \tag{1}$$

for some $r \in \mathbb{Z}^+$. The number r in (1) is called the *balancer* corresponding to the balancing number n. Behera and Panda also found that n is a balancing number if and only if n^2 is a triangular number, as well as, $8n^2 + 1$ is a perfect square. Though the definition suggests that no balancing number should be less than 2, we accept 1 as a balancing number being the positive square root of the square triangular number 1 [5]. If n is a balancing number then the positive root of $8n^2 + 1$ is called a *Lucas-balancing number* [6].

Let B_n and C_n denote the n^{th} balancing number and the n^{th} Lucas-balancing number, respectively, and set $B_0 = 0, C_0 = 1$. Then we have the recurrence relations

$$B_{n+1} = 6B_n - B_{n-1}, n \geq 1, \tag{2}$$

with $B_0 = 0, B_1 = 1$, and

$$C_{n+1} = 6C_n - C_{n-1}, n \geq 1, \tag{3}$$

with $C_0 = 1, C_1 = 3$. These recurrence relations give the Binet formulas for balancing and Lucas-balancing numbers as follows:

$$B_1 = 1, B_2 = 6, B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \text{ for all } n \geq 0, \tag{4}$$

and

$$C_1 = 3, C_2 = 17, C_n = \frac{\lambda_1^n + \lambda_2^n}{2}, \text{ for all } n \geq 0, \tag{5}$$

where $\lambda_1 = 3 + \sqrt{8}, \lambda_2 = 3 - \sqrt{8}$.

By slightly modifying (1), Panda and Ray [5] defined *cobalancing numbers* $n \in \mathbb{Z}^+$ as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r), \tag{6}$$

where r is called the *cobalancer* corresponding to n. An natural number n is a cobalancing number if and only if $8n^2+8n+1$ is a perfect square. So we can accept 0 is the first cobalancing number. Let b_n be the n^{th} cobalancing number. Then the n^{th} Lucas-cobalancing number c_n is the positive root of $8b_n^2 + 8b_n + 1$. Moreover, we have the recurrence relations [7]

$$b_1 = 0, b_2 = 2, b_{n+1} = 6b_n - b_{n-1} + 2, n \geq 2, \tag{7}$$

and

$$c_1 = 1, c_2 = 7, c_{n+1} = 6c_n - c_{n-1}, n \geq 2. \tag{8}$$

The following formulas are the Binet forms for cobalancing and Lucas-cobalancing numbers, respectively,

$$b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2} \text{ and } c_n = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2}, \tag{9}$$

where $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$.

Many interesting properties and important identities of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers are available in the literature. Panda [6] established two following identities which look like the trigonometric identities $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$:

$$B_{n+m} = B_n C_m + B_m C_n \tag{10}$$

and

$$B_{n-m} = B_n C_m - B_m C_n. \tag{11}$$

Before, some trigonometric-type identities have been established for Fibonacci numbers and Lucas numbers [8]. In this work, we establish some more trigonometric type identities and we deduce from these identities the parity of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers.

2 Sum of two balancing numbers

In this section, we prove some trigonometric-type identities of balancing number and deduce their parity. Starting from Panda's idea, the following theorem give an identity which has the same type of the trigonometric identity $\sin x - \sin y = 2\sin(\frac{x-y}{2})\cos(\frac{x+y}{2})$.

Theorem 2.1. *For n, m are natural numbers such that $n \geq m$ and having the same parity, we have*

$$B_n - B_m = 2B_{\frac{n-m}{2}} C_{\frac{n+m}{2}}. \tag{12}$$

Proof. Using Binet formulas, we have

$$\begin{aligned} 2B_{\frac{n-m}{2}} C_{\frac{n+m}{2}} &= 2.\frac{\lambda_1^{\frac{n-m}{2}} - \lambda_2^{\frac{n-m}{2}}}{\lambda_1 - \lambda_2}.\frac{\lambda_1^{\frac{n+m}{2}} + \lambda_2^{\frac{n+m}{2}}}{2} \\ &= \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} - \frac{\lambda_1^m - \lambda_2^m}{\lambda_1 - \lambda_2} \\ &= B_n - B_m. \end{aligned}$$

This completes the proof. □

Corollary 2.2. *For n, m are natural numbers such that $n \geq m$, we have*

$$B_{2n} - B_{2m} = 2B_{n-m} C_{n+m}. \tag{13}$$

Proof. This is an intermediate consequence of Theorem 2.1. □

In Corollary 2.2, by taking $m = 1$ we obtain a corollary of which [9, Theorem 2.1] is a particular case.

Corollary 2.3. *For $n \geq 1$, we have*

$$B_{2n} - 6 = 2B_{n-1}C_{n+1}. \tag{14}$$

With Theorem 2.1, we can see the parity of balancing numbers.

Theorem 2.4. *For every integer $n \geq 0$, the balancing number B_n and n have the same parity.*

Proof. If n, m are integers with the same parity then B_n and B_m have the same parity by Theorem 2.1. On the other one, we have $B_0 = 0, B_1 = 1$ and $B_2 = 6$. It implies that B_n and n have the same parity. □

We also have an identity of balancing numbers which resembles the trigonometric identity $\sin x + \sin y = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$.

Theorem 2.5. *Let n, m be natural numbers such that $n \geq m$ and having the same parity. Then*

$$B_n + B_m = 2B_{\frac{n+m}{2}}C_{\frac{n-m}{2}}. \tag{15}$$

Proof. Using Binet formulas, we have

$$\begin{aligned} 2B_{\frac{n+m}{2}}C_{\frac{n-m}{2}} &= 2.\frac{\lambda_1^{\frac{n+m}{2}} - \lambda_2^{\frac{n+m}{2}}}{\lambda_1 - \lambda_2}.\frac{\lambda_1^{\frac{n-m}{2}} + \lambda_2^{\frac{n-m}{2}}}{2} \\ &= \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} + \frac{\lambda_1^m - \lambda_2^m}{\lambda_1 - \lambda_2} \\ &= B_n + B_m. \end{aligned}$$

This is what was to be shown. □

3 Trigonometric-type identities of Lucas-balancing numbers

In the previous section, we have some trogonometric-type identities of balancing numbers in which balancing numbers are seen as sines while Lucas-balancing numbers are seen as cosines. Continuing with this point of view, we consider in this section trogonometric-type identities of Lucas-balancing numbers and their parity. The following theorem shows that we have an identity of Lucas-balancing numbers which looks like the trigonometric identity

$$\cos x + \cos y = 2\cos(\frac{x+y}{2})\cos(\frac{x-y}{2}). \tag{16}$$

However, we have another which resembles, up to a scalar, the trigonometric identity

$$\cos x - \cos y = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2}). \tag{17}$$

Theorem 3.1. *Let n, m be natural numbers such that $n \geq m$ and having the same parity. Then*

i) $C_n + C_m = 2C_{\frac{n+m}{2}} C_{\frac{n-m}{2}}$; (18)

ii) $C_n - C_m = 16B_{\frac{n+m}{2}} B_{\frac{n-m}{2}}$. (19)

Proof. Continue using Binet formulas, we have

$$\begin{aligned} 2C_{\frac{n+m}{2}} C_{\frac{n-m}{2}} &= 2.\frac{\lambda_1^{\frac{n+m}{2}} + \lambda_2^{\frac{n+m}{2}}}{2}.\frac{\lambda_1^{\frac{n-m}{2}} + \lambda_2^{\frac{n-m}{2}}}{2} \\ &= \frac{\lambda_1^n + \lambda_2^n}{2} + \frac{\lambda_1^m + \lambda_2^m}{2} = C_n + C_m. \end{aligned}$$

The first identity is proved. To prove the second, we have

$$\begin{aligned} B_{\frac{n+m}{2}} B_{\frac{n-m}{2}} &= \frac{\lambda_1^{\frac{n+m}{2}} - \lambda_2^{\frac{n+m}{2}}}{\lambda_1 - \lambda_2}.\frac{\lambda_1^{\frac{n-m}{2}} - \lambda_2^{\frac{n-m}{2}}}{\lambda_1 - \lambda_2} \\ &= \frac{1}{(\lambda_1 - \lambda_2)^2}(\lambda_1^n + \lambda_2^n - \lambda_1^m - \lambda_2^m) \\ &= \frac{1}{32}(\lambda_1^n + \lambda_2^n - \lambda_1^m - \lambda_2^m) = \frac{1}{16}(C_n - C_m). \end{aligned}$$

This implies the required identity. □

Corollary 3.2. *For n, m are natural numbers such that $n \geq m$, we have*

i) $C_{2n} + C_{2m} = 2C_{n+m}C_{n-m}$; (20)

ii) $C_{2n} - C_{2m} = 16B_{n+m}B_{n-m}$. (21)

Proof. These identities directly follow from Theorem 3.1. □

Now, we can see the parity of Lucas-balancing numbers.

Theorem 3.3. *For all integer $n \geq 0$, the Lucas-balancing number C_n is odd. Moreover, if n, m are integers with the same parity then the difference between C_n and C_m is divisible by 16.*

Proof. If n, m are integers with the same parity then the difference between C_n and C_m is divisible by 16 by the second identity of Theorem 3.1. This also means that C_n and C_m have the same parity. On the other hand, we have $C_0 = 1$,$C_1 = 3$, $C_2 = 17$. It implies that C_n is odd for all n. □

We can not find identities for Lucas-balancing numbers which resemble the trigonometric identities $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$. But we establish the following interesting identities.

Proposition 3.4. *Let n, m be natural numbers such that $n \geq m$. Then* *i)* $16(C_nC_m - B_nB_m) = 7C_{n+m} + 9C_{n-m}$; (22)

ii) $16(C_nC_m + B_nB_m) = 9C_{n+m} + 7C_{n-m}$. (23)

Proof. Applying Binet forms, we have

$$\begin{aligned} C_nC_m - B_nB_m &= \frac{\lambda_1^n + \lambda_2^n}{2}.\frac{\lambda_1^m + \lambda_2^m}{2} - \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.\frac{\lambda_1^m - \lambda_2^m}{\lambda_1 - \lambda_2} \\ &= \frac{\lambda_1^{n+m} + \lambda_2^{n+m} + \lambda_1^{n-m} + \lambda_2^{n-m}}{4} - \frac{\lambda_1^{n+m} + \lambda_2^{n+m} - \lambda_1^{n-m} - \lambda_2^{n-m}}{32} \\ &= \frac{7}{16}.\frac{\lambda_1^{n+m} + \lambda_2^{n+m}}{2} + \frac{9}{16}.\frac{\lambda_1^{n-m} + \lambda_2^{n-m}}{2} = \frac{7C_{n+m} + 9C_{n-m}}{16}. \end{aligned}$$

Hence we get the first identity. The second is proved by similar calculations. □

The following proposition give us relations between sums of Lucas-balancing numbers and Lucas-cobalancing numbers or cobalancing numbers from which we deduce an arithmetic property of sum of two consecutive Lucas-balancing numbers.

Proposition 3.5. *For n, m are integers such that $n \geq m \geq 1$, we have*

i) $C_{n+m-1} - C_{n-m} = 2c_nc_m$; (24)

ii) $C_{n+m-1} + C_{n-m} = 16b_nb_m + 8(b_n + b_m) + 4$. (25)

Proof. Using Binet forms with remark that $\alpha_1\alpha_2 = -1$, we have

$$\begin{aligned} c_nc_m &= \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2}.\frac{\alpha_1^{2m-1} + \alpha_2^{2m-1}}{2} \\ &= \frac{\alpha_1^{2(n+m-1)} + \alpha_2^{2(n+m-1)}}{2} - \frac{\alpha_1^{2(n-m)} + \alpha_2^{2(n-m)}}{2} \\ &= \frac{1}{2}(C_{n+m-1} - C_{n-m}). \end{aligned}$$

Hence we obtain the first identity. Similarly, we can prove the second identity. □

By ii) of Proposition 3.5, we have the following consequence about sum of two consecutive Lucas-balancing numbers.

Corollary 3.6. *For all integer $n \geq 1$, the sum of $(n-1)^{th}$ and n^{th} Lucas-balancing numbers is divisible by 4.*

4 Parity of cobalancing and Lucas-cobalancing numbers

In the last section, we establish some identities of cobalancing and Lucas-cobalancing numbers fromwhich we obtain some properties on the parity of these numbers. Firstly, we have an interesting property of sums of two cobalancing numbers and deduce the parity of cobalancing numbers.

Proposition 4.1. *Let n, m be positive integers.*

i) If $n > m$ then $b_{n+m} - b_{n-m} = 2c_n B_m$; (26)

ii) If $n \leq m$ then $b_{n+m} - b_{m-n+1} = 2c_n B_m$. (27)

Proof. By Binet formulas, we have

$$\begin{aligned} c_n B_m &= \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2} . \frac{\alpha_1^{2m} - \alpha_2^{2m}}{4\sqrt{2}} \\ &= \frac{\alpha_1^{2(n+m)-1} - \alpha_2^{2(n+m)-1}}{8\sqrt{2}} - \frac{\alpha_1^{2(n-m)-1} - \alpha_2^{2(n-m)-1}}{8\sqrt{2}} \\ &= \begin{cases} \frac{1}{2}(b_{n+m} - b_{n-m}), & \text{if } n > m; \\ \frac{1}{2}(b_{n+m} - b_{m-n+1}), & \text{otherwise.} \end{cases} \end{aligned}$$

Hence we have what was to be demonstrated. □

Theorem 4.2. *The cobalancing numbers are even. Moreover, for all $m \geq 1$, the difference between the $(2m+1)^{th}$ and $(2m)^{th}$ cobalancing numbers is divisible by 4.*

Proof. By ii) of Proposition 4.1, we can see that b_n and b_{n+1} have the same parity for all $n \geq 1$. It follows that b_n is even for all $n \geq 1$ since $b_1 = 0$. Moreover, from ii) of Proposition 4.1, we also obtain the second affirmation since B_{2m} is even by Theorem 2.4. □

In the following theorem, we again get an identity of cobalancing and Lucas-cobalancing numbers which looks like the trigonometric identity $\sin x + \sin y = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$.

Theorem 4.3. *Let n, m be positive integers.*

i) If $n > m$ then $c_{n+m} + c_{n-m} = 2c_n C_m$; (28)

ii) If $n \leq m$ then $c_{n+m} - c_{m-n+1} = 2c_n C_m$. (29)

Proof. By Binet forms, we have

$$\begin{aligned} c_n C_m &= \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2} . \frac{\alpha_1^{2m} + \alpha_2^{2m}}{2} \\ &= \frac{\alpha_1^{2(n+m)-1} + \alpha_2^{2(n+m)-1}}{4} + \frac{\alpha_1^{2(n-m)-1} + \alpha_2^{2(n-m)-1}}{4} \\ &= \begin{cases} \frac{1}{2}(c_{n+m} + c_{n-m}), & \text{if } n > m; \\ \frac{1}{2}(c_{n+m} - c_{m-n+1}), & \text{otherwise.} \end{cases} \end{aligned}$$

This completes the proof. □

We can deduce the parity of Lucas-cobalancing numbers from the second identity of the previous theorem.

Theorem 4.4. *The Lucas-cobalancing numbers are odd.*

Proof. By ii) of Theorem 4.3, we can see that c_n and c_{n+1} have the same parity for all $n \geq 1$. It follows that c_n is odd for all $n \geq 1$ since $c_1 = 1$. □

We finish with an identity from which we can see a better property on the parity of Lucas-cobalancing numbers of even index. It shows that the $(2n)^{th}$ Lucas-cobalancing number is congruent to -1 modulo 8 and the $(4n)^{th}$ Lucas-cobalancing number is congruent to -1 modulo 16, for all $n \geq 1$.

Proposition 4.5. *For integer $n \geq 1$, we have*

$$c_{2n} + 1 = 8(2b_n + 1)B_n. \tag{30}$$

Proof. By Binet forms, we have

$$\begin{aligned} b_n B_n &= \left(\frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2}\right) . \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} \\ &= \frac{\alpha_1^{4n-1} + \alpha_2^{4n-1}}{32} - \frac{\alpha_1^{-1} + \alpha_2^{-1}}{32} - \frac{1}{2}B_n \\ &= \frac{1}{16}(c_{2n} + 1) - \frac{1}{2}B_n. \end{aligned}$$

Hence we obtain the required identity. □

5 Conclusion

In this paper, we established some new trigonometric-type identities for balancing numbers, Lucas-balancing numbers, cobalancing numbers and Lucas-cobalancing numbers. Then we proved some arithmetic properties concerning the parity of these numbers.

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