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A METHOD OF APPROXIMATION FOR A ZERO OF MAXIMAL MONOTONE OPERATOR IN HILBERT SPACE

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ABSTRACT

In this paper, we introduce a new explicit iterative method for solving a variational inequality problem over the set of zeros for a maximal monotone operator in Hilbert space. By using two resolvents of the monotone operator at each iterate, we prove strong convergence of the method under a general condition on resolvent parameter.

Keywords: *Maximal monotone operators; Nonexpansive mappings; Fixed points; Zero points; Variational inequalities*

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MỘT PHƯƠNG PHÁP XẤP XỈ CHO KHÔNG ĐIỂM CỦA TOÁN TỬ ĐƠN ĐIỆU CỰC ĐẠI TRONG KHÔNG GIAN HILBERT

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TÓM TẮT

Trong bài báo này chúng tôi đưa ra một phương pháp lặp hiện mới giải bài toán bất đẳng thức biến phân trên tập không điểm của toán tử đơn điệu cực đại trong không gian Hilbert. Bằng việc sử dụng hai toán tử giải của một toán tử đơn điệu tại mỗi bước lặp, chúng tôi chứng minh sự hội tụ mạnh của phương pháp dưới điều kiện suy rộng đặt lên tham số.

Từ khóa: *Toán tử đơn điệu cực đại; ánh xạ không giãn; điểm bất động; không điểm; bất đẳng thức biến phân*

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1 Introduction

Let H be a real Hilbert space with inner product and norm denoted, respectively, by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let A be a maximal monotone operator in H. In this paper we assume that the set of zeros, $\Gamma := \{p \in \mathcal{D}(A) : \ 0 \in Ap\}$, is nonempty, where $\mathcal{D}(A)$ denotes the domain of A.

Finding a zero of a maximal monotone operator, i.e., finding a point

$$finding\ a\ point \quad p \in \mathcal{D}(A) : \ such\ that \quad 0 \in Ap, \tag{1.1}$$

is an important part of the theory of monotone operators. A fundamental method for finding a zero point of a maximal monotone operator A in Hilbert space H, we can cite the proximal point one [1]. This method generates a sequence $\{x^k\}$ defined for each $k \geq 1$

$$x^{k+1} = J_k^A x^k + e^k \quad or \quad x^{k+1} = J_k^A (x^k + e^k) \tag{1.2}$$

where $x^1 \in H$, $J_k^A = (I + r_k A)^{-1}$, I is the identity mapping of H, $\{r_k\}$ is a sequence of real numbers such that $r_k \geq \varepsilon > 0$ for all $k \geq 1$ and e^k is an error vector, satisfying

$$\|x^{k+1} - J_k^A x^k\| \leq \varepsilon_k \quad with \quad \sum_{k=1}^{\infty} \varepsilon_k < \infty$$

or

$$\|x^{k+1} - J_k^A x^k\| \leq \eta_k \|x^{k+1} - x^k\| \quad with \quad \sum_{k=1}^{\infty} \eta_k < \infty. \tag{1.3}$$

Methods (1.2)-(1.3) converge only weakly to a zero of A in infinite-dimensional Hilbert spaces, in general (see, [2]). In order to have a strong convergence sequence $\{x^k\}$ from the method, several modifications of (1.2) were proposed in [3–5]. Kamimura and Takahashi [3] introduced a method, in there two sequences $\{x^k\}$ and $\{y^k\}$ are built from a starting point x^1 as follows:

$$y^k \approx J_k^A x^k, \quad \|y^k - J_k^A x^k\| \leq \delta_k, \quad x^{k+1} = t_k u + (1 - t_k) y^k, \quad k \geq 1, \tag{1.4}$$

where u is a fixed point in H. The sequence $\{x^k\}$ so generated, as $k \to \infty$, is strongly convergent to $P_\Gamma u$, the metric projection of u on the set Γ, under the following conditions:

(C1) $t_k \in (0, 1)$ for all $k \geq 1$, $\lim_{k\to\infty} t_k = 0$ and $\sum_{k=1}^{\infty} t_k = \infty$;

(C2) $r_k \in (0, \infty)$ for all $k \geq 1$ and $\lim_{k\to\infty} r_k = \infty$; and

(C3) $\sum_{k=1}^{\infty} \delta_k < \infty$.

Xu [5] extended the prox-Tikhonov method of Lehdili and Moudafi [4] in the following way

$$x^{k+1} = J_k^A(t_k u + (1-t_k)x^k + e^k). \tag{1.5}$$

Further, Boikanyo and Morosanu [6] showed that (1.5) is equivalent to

$$y^{k+1} = t_k u + (1-t_k)J_k^A y^k + e^k, \tag{1.6}$$

and proved a strong convergence result for $\{y^k\}$, defined by (1.6), to $P_\Gamma u$, if there hold conditions (C1), (C2) and

(C3') either (C3) with $\delta_k = \|e^k\|$ or $\lim_{k\to\infty}(\|e_k\|/t_k) = 0$.

In this paper, we introduce a new modifications of (1.2),

$$x^{k+1} = J_k^A J_c^A(t'_k u + (1-t'_k)x^k) + e^k \tag{1.7}$$

where $J_c^A = (I + cA)^{-1}$ and c is any fixed positive real number. We will show that method (1.7) is particular case of the following method,

$$z^{k+1} = J_k^A J_c^A(I - t_k\mu F)z^k + e^k, \tag{1.8}$$

proposed to solve a problem of finding a point

$$p_* \in \Gamma \quad \textit{such that} \quad \langle Fp_*, p_* - p\rangle \le 0 \quad \forall p \in \Gamma, \tag{1.9}$$

where $\mu \in (0, 2\eta/L^2)$ is a constant and $F : H \to H$ is an η-strongly monotone and L-Lipschitz continuous operator with η, $L > 0$.

The paper is organized as follows. In Section 2, we list some related facts that will be used in our result. In Section 3, we prove strong convergence of our main method and show that their special case is new contraction and generalized proximal point method, that converge strongly to a zero under a general condition on the resolvent parameter.

2 Preliminaries

In this section, we introduce some mathematical symbols, definitions, and lemmas which can be used in the proof of our main result.

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. In what follows, we write $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}$ converges weakly to x while $x^k \to x$ indicate that the sequence $\{x^k\}$ converges strongly to x.

First, we know that, for any Hilbert space H,

$$\|x+y\|^2 \leq \|x\|^2 + \ 2\langle y, x+y\rangle \quad \forall x, y \in H.$$

Let C be a nonempty, closed and convex subset of H. We know that, for each $x \in H$, there is a unique $P_C x \in C$ such that

$$\|x - P_C x\| = \inf_{u\in C} \|x-u\|, \tag{2.1}$$

and the mapping $P_C :\ H \to C$ defined by (2.1) is called the metric projection from H onto C. Moreover, we have

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall x \in H,\ y \in C, \tag{2.2}$$

(see, for example, [7, Section 3]).
Let $F : H \to H$ be a mapping. F is said to be L-Lipschitz continuous and η-strongly monotone when the following conditions are satisfied:

$$\|Fx - Fy\| \leq L\|x-y\| \quad \textit{and} \quad \langle Fx - Fy, x-y\rangle \geq \eta\|x-y\|^2$$

for all $x, y \in H$, where L and η are some positive constants. F is said to be contraction operator, if $0 \leq L < 1$ and nonexpansive, if $L = 1$.

Lemma 2.1 [see, [8]]Let H be a real Hilbert space and let F be an η-strongly monotone and L-Lipschitz continuous operator on H with some positive constants η and L. Then, for a fixed number $\mu \in (0, 2\eta/L^2)$ and any $t \in (0,1)$, $I - t\mu F$ is a contraction with contractive constant $1 - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$.

We introduce some definitions and propositions about set–valued mappings. Let A be a set–valued operator of H into 2^H with domain $\mathcal{D}(A) = \{x \in H :\ \ A(x) \neq \emptyset\}$, range $\mathcal{R}(A) = \cup_{x\in\mathcal{D}(A)} Ax$, and the inverse of A is $A^{-1}(y) = \{x \in H : y \in A(x)\}$. $A : H \to 2^H$ is said to be

(i) monotone if $\langle u - v, x - y\rangle \geq 0\ \forall x, y \in \mathcal{D}(A),\ u \in A(x),\ v \in A(y)$;

(ii) maximal monotone if it is monotone and the graph

$$\mathcal{G}(A) = \{(x, y) \in H \times H :\ x \in \mathcal{D}(A),\ y \in A(x)\}$$

of A is not properly contained in the graph of any other monotone operator on $\mathcal{D}(A)$.

For a monotone mapping A, we define its resolvent J_r^A by

$$J_r^A := (I + rA)^{-1} : \mathcal{R}(I + rA) \subset H \to \mathcal{D}(A),$$

where $r > 0$, I is the identity operator on H.
A fixed point of the mapping $F : C \to C$ is a point $x \in C$ such that $Fx = x$. The set of all fixed points of the mapping F is denoted by $\text{Fix}(F)$.

Lemma 2.2 [9, Section 7]Let H be a real Hilbert space. If $A : H \to H$ is a maximal monotone operator,

(i) J_r^A is nonexpansive, single-valued mapping and $\text{Fix}(J_r^A) = A^{-1}(0)$ for each $r > 0$, and

(ii)
$$\|J_r^A x - p\|^2 \leq \|x - p\|^2 - \|J_r^A x - x\|^2 \quad \forall x \in H,\ p \in A^{-1}(0).$$

Lemma 2.3 [see, [10]] Let $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ be sequences of real numbers such that, for all $k \geq 1$, $a_{k+1} \leq (1 - b_k)a_k + b_k c_k$; $a_k \geq 0$;

(i) b_k satisfies a condition of type **(T)**; and

(ii) either $\sum_{k=1}^{\infty} b_k|c_k| < \infty$ or $\limsup_{k\to\infty} c_k \leq 0$.

Then, $lim_{k\to\infty} a_k = 0$.

Lemma 2.4 [see, [11]] Let $\{a_k\}$ be a sequence of real numbers with a subsequence $\{k_l\}$ of $\{k\}$ such that $a_{k_l} < a_{k_l+1}$ for all $l \in N_+$. Then, there exists a nondecreasing sequence $\{m_k\} \subseteq N_+$ such that $m_k \to \infty, a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$ for all (sufficiently large) numbers $k \in N_+$. In fact, $m_k = \max\{l \leq k : a_l \leq a_{l+1}\}$.

Proposition 2.1 [see, [12,13]] Let H and F be as in Lemma 2.1 and let T be a nonexpansive operator on H such that $\text{Fix}(T) \neq \emptyset$. Then, for any bounded sequence $\{x^k\} \subset H$ such that $\lim_{k\to\infty} \|Tx^k - x^k\| = 0$, we have

$$\limsup_{k\to\infty} \langle Fp_*, p_* - x^k \rangle \leq 0, \tag{2.3}$$

where $p_* \in \text{Fix}(T)$, solving (1.9) with Γ replaced by $\text{Fix}(T)$.

3 Main Results

First of all, we have the following results.

Theorem 3.1 Let A be a maximal monotone operator in a real Hilbert space H such that $\Gamma := \{p \in \mathcal{D}(A) : 0 \in Ap\} \neq \emptyset$ and let F with μ be as in Lemma 2.1. Assume that there hold conditions (C1), (C3'), and

(C2') $\{r_k\}$ is any sequence of numbers in $(0, \infty)$.

Then, the sequence $\{z^k\}$, defined by (1.8), as $k \to \infty$, converges strongly to the unique solution p_*, solving (1.9).

Proof We consider an exact variant of (1.8), that is,

$$x^{k+1} = J_k^A J_c^A (I - t_k \mu F) x^k. \tag{3.1}$$

Clearly, from (1.8), (3.1), the nonexpansive property of J_k^A and Lemma 2.1, we get the following inequality:

$$\|z^{k+1} - x^{k+1}\| = \|J_k^A J_c^A (I - t_k \mu F) z^k + e^k - J_k^A J_c^A (I - t_k \mu F) x^k\|$$

$$\leq (1 - t_k \tau) \|z^k - x^k\| + \|e^k\|.$$

According to conditions (C1) and (C3'), we apply Lemma 2.2 to conclude that $\|z^k - x^k\| \to 0$ as $k \to \infty$. Hence, to show the desired result, it suffices to prove that $\{x^k\}$ converges strongly to p_* as $k \to \infty$. For this purpose, we first prove that $\{x^k\}$ is bounded. Indeed, for a fixed point $p \in \Gamma$, by Lemma 2.1, we have

$$\|x^{k+1} - p\| = \|J_k^A J_c^A (I - t_k \mu F) x^k - J_k^A J_c^A p\| \leq (1 - t_k \tau) \|x_k - p\| + t_k \mu \|Fp\|$$

$$\leq \max\{\|x^1 - p\|, \mu \|Fp\|/\tau\},$$

by mathematical induction. Therefore, the sequence $\{x^k\}$ is bounded, and so are the sequences $\{Fx^k\}$ and $\{y^k\}$ where $y^k := (I - t_k \mu F) x^k$. Without loss of generality, we can assume that they are bounded by a positive constant M_1.

Further, we estimate the value $\|x^{k+1} - p\|^2$ as follows.

$$\|x^{k+1} - p\|^2 = \|J_k^A J_c^A y^k - J_k^A p\|^2 \leq \|J_c^A y^k - p\|^2 \leq \|y^k - p\|^2 - \|J_c^A y^k - y^k\|^2$$

$$= \|(I - t_k \mu F) x^k - p\|^2 - \|J_c^A y^k - y^k\|^2 \leq (1 - t_k \tau) \|x^k - p\|^2 + 2 t_k \mu \langle Fp, p - y^k \rangle - \|J_c^A y^k - y^k\|^2. \tag{3.2}$$

We need only consider two cases.

Case 1. There exists an integer $k_0 \geq 1$ such that $\|x^{k+1} - p\| \leq \|x^k - p\|$ for all $k \geq k_0$.

Then, $\lim_{k\to\infty} \|x^k - p\|$ exists. From (3.2), we can write that

$$\|J_c^A y^k - y^k\|^2 \leq \|x^k - p\|^2 - \|x^{k+1} - p\|^2 + 2 t_k \mu M \tag{3.3}$$

where $M \geq \|Fp\|(\|p\| + M_1)$. Since $\lim_{k\to\infty} \|x^k - p\|$ exists and $t_k \to 0$, letting k tend to infinity in (3.3), we get that $\lim_{k\to\infty} \|J_c^A y^k - y^k\| = 0$. This together with $\|y^k - x^k\| \leq t_k \mu M_1$ implies that $\lim_{k\to\infty}\|J_c^A x^k - x^k\| = 0$. By using Proposition 2.1 with $T = J_c^A$, we obtain inequality (2.3). Now, from (3.2) with $p = p_*$, we know that

$$\|x^{k+1} - p_*\|^2 \le (1 - t_k\tau)\|x^k - p_*\|^2 + 2t_k\mu\langle Fp_*, p_* - x^k + t_k\mu Fx^k\rangle$$
$$= (1 - b_k)\|x^k - p_*\|^2 + b_k c_k \quad for\ all \quad k \ge k_0,$$

where $b_k = t_k\tau$ and $c_k = \frac{2\mu}{\tau}[\langle Fp_*, p_* - x^k\rangle + t_k\mu\|Fp_*\|M_1]$, from which and Lemma 2.2 we obtain that $\|x^k - p_*\| \to 0$.

Case 2. There exists a subsequence $\{k_l\}$ of $\{k\}$ such that $\|x^{k_l} - p\| < \|x^{k_l+1} - p\|$ for all $l \in N_+$. Hence, by Lemma 2.4, there exists a nondecreasing sequence $\{m_k\} \subseteq N_+$ such that $m_k \to \infty$,

$$\|x^{m_k} - p\| \le \|x^{m_k+1} - p\| \quad and \quad \|x^k - p\| \le \|x^{m_k+1} - p\| \tag{3.4}$$

for each $k \in N_+$. Then, from (3.2) and the first inequality in (3.4), we know that

$$\|x^{m_k} - p\|^2 \le \frac{2\mu}{\tau}\langle Fp, p - y^{m_k}\rangle. \tag{3.5}$$

On the other hand, again from (3.2) the first inequality in (3.4), we have also that

$$\|J_c^A y^{m_k} - y^{m_k}\|^2 \le 2t_{m_k}\mu M.$$

Therefore, $\lim_{k\to\infty}\|J_c^A y^{m_k} - y^{m_k}\| = 0$, and hence, by Proposition 2.1,

$$\limsup_{k\to\infty}\langle Fp_*, p_* - y^{m_k}\rangle \le 0,$$

from which and (3.5) with p replaced by p_*, it follows that

$$\lim_{k\to\infty}\|x^{m_k} - p_*\| = 0. \tag{3.6}$$

Finally, from (3.2) with k and p replaced, respectively, by m_k and p_*, we can write that

$$\|x^{m_k+1} - p_*\|^2 \le (1 - t_{m_k}\tau)\|x^{m_k} - p_*\|^2 + 2t_{m_k}\mu\langle Fp_*, p_* - y^{m_k}\rangle.$$

By virtue of (3.6) and $t_{m_k} \to 0$, $\lim_{m\to\infty}\|x^{m_k+1} - p_*\|^2 = 0$, which together with the second inequality in (3.4) implies that $\lim_{k\to\infty}\|x^k - p_*\| = 0$. This completes the proof.

Remark 3.1 Clearly, the mapping $F = I - f$, where $f = \tilde{a}I + (1 - \tilde{a})u$ with a fixed number $\tilde{a} \in (0, 1)$ and a fixed point $u \in H$, is an η-strongly monotone and L-Lipschitz continuous operator with $\eta = 1 - \tilde{a}$ and $L = 1 + \tilde{a}$. Then, replacing F in (1.8) by $I - f$ and re-denoting $z^k := x^k$ with $t'_k := t_k\mu(1 - \tilde{a})$, we obtain (1.7). Now, putting $y^k = (I - t_k\mu F)x^k$ in (1.8), we get that

$$y^{k+1} = (I - t_{k+1}\mu F)x^{k+1} = (I - t_{k+1}\mu F)(J_k^A J_c^A y^k + e^k).$$

4 Conclusion

We have presented an iterative method for finding a point in the zero set of a maximal monotone mapping in Hilbert space, that solves a variational inequality problem, involving an η-strongly monotone and L-Lipschitz continuous operator on H for some positive constants η and L. As consequences, new generalized and contraction proximal point algorithm with a any sequence of positive numbers for the resolvent parameter have been obtained.

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