ON HYPERSTABILITY OF GENERALIZED LINEAR EQUATIONS IN SEVERAL VARIABLES IN QUASI-NORMED SPACES

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Abstract

In this paper, we state and prove the hyperstability of generalized linear equations in several variables in quasi-normed spaces. As applications, we deduce some known results and some particular cases of generalized linear equations in several variables.

Keywords: *Fixed point; linear equations in several variables; quasi-normed space.*

THIẾT LẬP TÍNH SIÊU ỔN ĐỊNH CỦA PHƯƠNG TRÌNH HÀM TUYẾN TÍNH SUY RỘNG NHIỀU BIẾN TRONG KHÔNG GIAN TỰA CHUẨN

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Tóm tắt

Trong bài báo này, chúng tôi thiết lập và chứng minh tính siêu ổn định của phương trình hàm tuyến tính suy rộng nhiều biến trong không gian tựa chuẩn. Đồng thời, sử dụng kết quả đạt được, chúng tôi suy ra một số kết quả đã có và một số trường hợp đặc biệt của lớp phương trình hàm tuyến tính suy rộng nhiều biến.

Từ khóa: *Điểm bất động, không gian tựa chuẩn, phương trình hàm tuyến tính suy rộng nhiều biến.*

1. Introduction

Studies of the stability of functional equations date back to (Hyers, 1941) and (Ulam, 1964). A particular case of the stability problem that is of interest to some authors is the hyperstability of linear functional equations. Results on the hyperstability have first appeared in (Bourgi, 1949), but the term "hyperstability" was first used in (Maksa and Pales, 2001). Some authors have studied the Hyers-Ulam's hyperstability for various classes of linear functional equations. Recently, some authors have studied the class of generalized linear functional equations with many variables of the form

$$\sum_{i=1}^{m} L_i f\left(\sum_{j=1}^{n} a_{ij} x_j\right) = 0. \qquad (1.1)$$

In 2015, the result of equation (1.1) was established and proved (Zhang, 2015). Specifically, with appropriate assumptions, the approximate solution of the generalized linear functional equation (1.1) is the solution of that equation. The main way to the proof of the paper (Zhang, 2015) is to use Brzdek's fixed point theorem (Brzdek *et al.*, 2011). Besides, the normed space has been expanded into a quasi-normed space with many different characterizations. Dung and Hang (2018) established a fixed point theorem in the quasi-normed space and applied it to study the hyperstability of functional equations in quasi-Banach space.

In this paper, we use the fixed point theorem in Dung and Hang (2018) to establish and prove the hyperstability of generalized linear equations in several variables in quasi-normed spaces.

Now we recall some notions.

Definition 1.1 (Kalton, 2003, p. 1102). Let X be a vector space over the field $\mathbb{K}$, $\kappa \geq 1$ and $\|.\|: X \to \mathbb{R}_+$ be a fuction such that for all $x, y \in X$ and all $a \in \mathbb{K}$,

1. $\|x\| = 0$ if and only if $x = 0$.
2. $\|ax\| = |a|.\|x\|$.
3. $\|x+y\| \leq \kappa(\|x\| + \|y\|)$.

Then

1. $\|.\|$ is called a *quasi-norm* on X and $(X, \|.\|, \kappa)$ is called a *quasi-normed space*.
2. $\|.\|$ is called a p*-norm* on X and $(X, \|.\|, \kappa)$ is called a p*-normed space* if there is $0 < p \leq 1$ such that

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p \text{ for all } x, y \in X.$$

3. The sequence $\{x_n\}_n$ is called *convergent* to x if $\lim_{n\to\infty} \|x_n - x\| = 0$, denoted by $\lim_{n\to\infty} x_n = x$.
4. The sequence $\{x_n\}_n$ is called *Cauchy* if $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$.
5. The quasi-normed space $(X, \|.\|, \kappa)$ is called *quasi-Banach* if each Cauchy sequence is a convergent sequence.
6. The quasi-normed space $(X, \|.\|, \kappa)$ is called p*-Banach* if it is p-norm and quasi-Banach.

Remark 1.2.

1. If $\kappa = 1$ then a quasi-normed space is a normed space.
2. p-norm is a continuous function.
3. For all $x_1, \ldots, x_n \in X$ we have $\left\|\sum_{i=1}^{n} x_i\right\| \leq \kappa^{n-1} \sum \|x_i\|$.

Example 1.3 (Kalton *et al.*, 1984, p. 17). The space

$$L_p[0,1] = \left\{x : [0,1] \to \mathbb{R} : \int_{[0,1]} |x(t)|^p \, dt < \infty\right\}$$

where $0 < p \leq 1$, $\|x\| = \left(\int_{[0,1]} |x(t)|^p \, dt\right)^{\frac{1}{p}}$ for all $x \in L_p[0,1]$ is quasi-normed space with $\kappa = 2^{\frac{1}{p}-1}$.

The following corollary is used to study the hyperstability of generalized linear equations in several variables in quasi-Banach spaces.

Corollary 1.4 (Dung and Hang, 2018, Corollary 2.2)**.** *Suppose that*

1. U is a nonempty set, $(Y,\|.\|,\kappa)$ is a quasi-Banach space, and $\mathcal{T}: Y^U \to Y^U$ is a given function, Y^U is the set of all mappings from U to Y.

2. There exist $f_1,\dots,f_k : U \to U$ and $L_1,\dots,L_k : U \to \mathbb{R}_+$ such that for all $\xi,\mu \in Y^U$ and $x \in U$,

$$\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x)\|$$

$$\leq \sum_{i=1}^{k} L_i(x)\|\xi\big(f_i(x)\big)-\mu\big(f_i(x)\big)\|. \tag{1.2}$$

3. There exist $\varepsilon: U \to \mathbb{R}_+$ and $\varphi: U \to Y$ such that for all $x \in U$,

$$\|\mathcal{T}\varphi(x)-\varphi(x)\| \leq \varepsilon(x). \tag{1.3}$$

4. For every $x \in U$ and $\theta = \log_{2\kappa} 2$,

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)^\theta (x) < \infty \tag{1.4}$$

where $\Lambda: \mathbb{R}_+^U \to \mathbb{R}_+^U$ defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta\big(f_i(x)\big) \tag{1.5}$$

for all $\delta: U \to \mathbb{R}_+$ and $x \in U$.

Then we have,

1. For every $x \in U$, the limit

$$\lim_{n\to\infty} \mathcal{T}^n \varphi(x) = \psi(x) \tag{1.6}$$

exists and the so difined function $\psi: U \to Y$ is a fixed point of $\mathcal{T}$ satisfying

$$\|\varphi(x)-\psi(x)\|^\theta \leq 4\varepsilon^*(x) \tag{1.7}$$

for all $x \in U$.

2. For every $x \in U$, if there exists a positive real M such that

$$\varepsilon^*(x) = \Big(M\sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x)\Big)^\theta < \infty \tag{1.8}$$

then the fixed point of $\mathcal{T}$ satisfying (1.7) is unique.

The following result is well-known and is usually called Aoki-Rolewicz theorem.

Theorem 1.5 (Maligranda, 2008, Theorem 1)**.** *Let $(X,\|.\|,\kappa)$ be a quasi-normed space, $p = \log_{2\kappa} 2$ and $|||.|||: X \to \mathbb{R}_+$ defined by*

$$|||x||| = \inf\left\{\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{\frac{1}{p}} : x = \sum_{i=1}^{n} x_i : x_i \in X, n \geq 1\right\}$$

for all $x \in X$. Then $|||.|||$ is p-norm on X and

$$\frac{1}{2\kappa}\|x\| \leq |||x||| \leq \|x\|, \text{ for all } x \in X.$$

2. Main results

In this section, we establish and prove some results on the hyperstability of the generalized linear equations in several variables (1.1) in quasi-normed spaces.

Theorem 2.1. *Suppose that*

1. $\mathbb{F},\mathbb{K}$ denote the fields of real or complex numbers and $(X,\|.\|_X,\kappa_X)$ is a quasi-normed space over field $\mathbb{F}$, $(Y,\|.\|_Y,\kappa_Y)$ is a quasi-Banach space over field $\mathbb{K}$ and $f: X \to Y$ is a given mapping.

2. $n \geq 2$ and m are positive integers, $C \geq 0$, $a_{ij} \in \mathbb{F}$ and $L_i \in \mathbb{K}$ are given parameters for $i = 1,\dots,m$, $j = 1,\dots,n$.

3. There exist $i_0 \in \{1,\dots,m\}$ and $j_1 \neq j_2 \in \{1,\dots,n\}$ such that $a_{i_0 j_1} \neq 0$, $a_{i_0 j_2} \neq 0$. For all $i \neq i_0$, $\gamma \neq 0$, there is $j \in \{1,\dots,n\}$ satisfying $a_{ij} \neq \gamma a_{i_0 j}$.

4. *There exists* $p<0$ *such that*

$$\left\|\sum_{i=1}^{m} L_i f\left(\sum_{j=1}^{n} a_{ij}x_j\right)\right\|_Y \le C\sum_{j=1}^{n}\| x_j \|_X^p \quad (2.1)$$

for all $x_1,\ldots,x_n \in X\setminus\{0\}$.

Then we have

$$\sum_{i=1}^{m} L_i f\left(\sum_{j=1}^{n} a_{ij}x_j\right)=0 \quad (2.2)$$

for all $x_1,\ldots,x_n \in X\setminus\{0\}$.

Proof. Without any loss of generality, we may assume that $i_0=1$ and $(a_{1j})_{1\times n}$ is the row satisfying Condition (3). For $i=1,\ldots,m,$ let π_i denote the hyperplane $\sum_{j=1}^{n} a_{ij}t_j=0$ in $\mathbb{F}^n$. For $k=1,\ldots,n,$ let $\pi_{c,k}$ be the coordinate plane $t_k=0$ in $\mathbb{F}^n$. Then π_1 is the hyperplane $\sum_{j=1}^{n} a_{1j}t_j=0.$ By the hypothesis on $(a_{1j})_{1\times n}$, it follows that $\pi_1\neq\pi_i$ $(i=2,\ldots,m)$ and $\pi_1\neq\pi_{c,k}$. So, we get

$$\left(\pi_1\setminus\bigcup_{k=1}^{n}\pi_{c,k}\right)\setminus\bigcup_{i=2}^{m}\pi_i\neq\varnothing.$$

Choose an element

$$(k_1,\ldots,k_n)\in\left(\pi_1\setminus\bigcup_{k=1}^{n}\pi_{c,k}\right)\setminus\bigcup_{i=2}^{m}\pi_i.$$

Obviously, $(k_1,\ldots,k_n)$ satisfies

$$\begin{cases}\sum_{j=1}^{n} a_{1j}k_j=0\\ k_j\neq 0,\ j=1,\ldots,n\\ \sum_{j=1}^{n} a_{ij}k_j\neq 0, i=2,\ldots,m.\end{cases}$$

Keep the hypothesis on $(a_{1j})_{1\times n}$ in mind, there exists $b_1,\ldots,b_n\in\mathbb{F}$ such that $\sum_{j=1}^{n} a_{1j}b_j=1.$

For a given large $t\in\mathbb{Z}_+$, $(k_jt+b_j)\neq 0$ and $x\neq 0,$ we set $x_j=(k_jt+b_j)x$, $j=1,\ldots,n$, and write $s_i(t)=\sum_{j=1}^{n} a_{ij}(k_jt+b_j)$, $i=1,\ldots,m.$ Then

$$s_1(t)=\sum_{j=1}^{n} a_{1j}(k_jt+b_j)=\sum_{j=1}^{n} a_{1j}k_jt+\sum_{j=1}^{n} a_{1j}b_j=1$$

and the inequality (2.1) takes the form

$$\left\|\sum_{i=1}^{m} L_i f\big(s_i(t)x\big)\right\|_Y \le C\sum_{j=1}^{n}|k_jt+b_j|^p\| x \|_X^p . \quad (2.3)$$

From (2.3), we gain

$$\left\|L_1 f(x)+\sum_{i=2}^{m} L_i f\big(s_i(t)x\big)\right\|_Y \le C\sum_{j=1}^{n}|k_jt+b_j|^p\| x \|_X^p .$$

Dividing the two sides of the above inequality by $|-L_1|,$ we obtain

$$\left\|-1.f(x)+\sum_{i=2}^{m}\frac{L_i}{-L_1} f\big(s_i(t)x\big)\right\|_Y$$

$$\le\frac{C}{|-L_1|}\sum_{j=1}^{n}|k_jt+b_j|^p\| x \|_X^p,$$

set $L_1:=-1,$ $L_i:=\dfrac{L_i}{-L_1}$ and $C:=\dfrac{C}{|-L_1|}$. Then we can use (2.1) as the form

$$\left\|\sum_{i=2}^{m} L_i f\big(s_i(t)x\big)-f(x)\right\|_Y \le C\sum_{j=1}^{n}|k_jt+b_j|^p\| x \|_X^p . \quad (2.4)$$

Since $k_1,\ldots,k_n\neq 0$ we have $\lim_{t\to\infty}\left|k_jt+b_j\right|=+\infty,$ for all $j=1,\ldots,n.$ Define $\alpha_t:=C\sum_{j=1}^{n}|k_jt+b_j|^p,$ so that

$$\lim_{t\to\infty}\alpha_t=0. \quad (2.5)$$

We can suppose that t is sufficiently large so that $0\le\alpha_t<1.$

Define mapping $T_t: Y^{X\setminus\{0\}}\to Y^{X\setminus\{0\}}$ by

$$T_t\xi(x)=\sum_{i=2}^{m} L_i\xi\big(s_i(t)x\big)$$

for all $x \in X \setminus \{0\}$ and $\xi \in Y^{X\setminus\{0\}}$. We set

$$\varepsilon_t(x) = \alpha_t \| x \|_X^p \tag{2.6}$$

for all $x \in X \setminus \{0\}$. The inequality (2.4) can be written as

$$\| T_t f(x) - f(x) \|_Y \le \varepsilon_t(x).$$

This proves that (1.3) is satisfied.

Define mapping $\Lambda_t : \mathbb{R}_+^{X\setminus\{0\}} \to \mathbb{R}_+^{X\setminus\{0\}}$ by

$$\Lambda_t \delta(x) = \kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| \, \delta\big(s_i(t)x\big) \tag{2.7}$$

for all $x \in X \setminus \{0\}$ and $\delta \in \mathbb{R}_+^{X\setminus\{0\}}$. This proves that (2.7) has the form as (1.5), where L_i is replaced by $\kappa_Y^{m-2} |L_i|$. Furthermore, for all $\xi, \eta \in Y^{X\setminus\{0\}}$, $x \in X \setminus \{0\}$ and Remark 1.2 (3), we have

$$\| T_t\xi(x) - T_t\eta(x) \|_Y$$

$$= \left\| \sum_{i=2}^{m} L_i \xi\big(s_i(t)x\big) - \sum_{i=2}^{m} L_i \eta\big(s_i(t)x\big) \right\|_Y$$

$$= \left\| \sum_{i=2}^{m} \big[L_i \xi\big(s_i(t)x\big) - L_i \eta\big(s_i(t)x\big) \big] \right\|_Y$$

$$\le \kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . \| \xi\big(s_i(t)x\big) - \eta\big(s_i(t)x\big) \|_Y .$$

It proves that (1.2) is satisfied when L_i is replaced by $\kappa_Y^{m-2} |L_i|$.

For all $x \in X \setminus \{0\}$ we have

$$\Lambda_t \varepsilon_t(x)$$

$$= \kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . \alpha_t \| s_i(t)x \|_X^p$$

$$= \kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . |s_i(t)|^p \, \alpha_t \| x \|_X^p .$$

By induction, we will show that for all $x \in X \setminus \{0\}$, $n \in \mathbb{N}$

$$\Lambda_t^n \varepsilon_t(x) = \left(\kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . |s_i(t)|^p \right)^n \alpha_t \| x \|_X^p . \tag{2.8}$$

Indeed, if $n = 0$, then (2.8) holds by (2.6). Suppose that (2.8) holds for $n = k$, that is,

$$\Lambda_t^k \varepsilon_t(x) = \left(\kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . |s_i(t)|^p \right)^k \alpha_t \| x \|_X^p .$$

We have

$$\Lambda_t^{k+1} \varepsilon_t(x)$$

$$= \Lambda_t \big(\Lambda_t^k \varepsilon_t(x) \big)$$

$$= \kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| \left(\kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . |s_i(t)|^p \right)^k \alpha_t \| s_i(t)x \|_X^p$$

$$= \left(\kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . |s_i(t)|^p \right) . \left(\kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . |s_i(t)|^p \right)^k \alpha_t \| x \|_X^p$$

$$= \left(\kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . |s_i(t)|^p \right)^{k+1} \alpha_t \| x \|_X^p .$$

So, (2.8) holds for all $n \in \mathbb{N}$.

By using (2.8) with $\theta = \log_{2\kappa_Y} 2$, we gain

$$\varepsilon_t^*(x)$$

$$= \sum_{n=0}^{\infty} (\Lambda_t^n \varepsilon_t)^\theta (x)$$

$$= \sum_{n=0}^{\infty} \left(\kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . |s_i(t)|^p \right)^{\theta n} \alpha_t^\theta \| x \|_X^{\theta p} . \tag{2.9}$$

$s_i(t) = \sum_{j=1}^{n} a_{ij}(k_j t + b_j) = \sum_{j=1}^{n} a_{ij} k_j t + \sum_{j=1}^{n} a_{ij} b_j$ and $\sum_{j=1}^{n} a_{ij} k_j \ne 0$ for all $i = 2, \ldots, m$, we have

$$\lim_{t\to\infty} |s_i(t)| = \lim_{t\to\infty} \left| \sum_{j=1}^{n} a_{ij} k_j t + \sum_{j=1}^{n} a_{ij} b_j \right| = +\infty.$$

So, we gain $\lim_{t\to\infty} \kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . |s_i(t)|^p = 0$. We choose a large positive integer t such that

$$\kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . |s_i(t)|^p < 1. \tag{2.10}$$

Then

$$\sum_{n=0}^{\infty}\left(\kappa_Y^{m-2}\sum_{i=2}^{m}|L_i|.|s_i(t)|^p\right)^{\theta n}$$
$$=\frac{1}{1-\left(\kappa_Y^{m-2}\sum_{i=2}^{m}|L_i|.|s_i(t)|^p\right)^{\theta}}. \qquad (2.11)$$

By using (2.9) and (2.11), we have

$$\varepsilon_t^*(x)=\frac{\alpha_t^{\theta}\|x\|_X^{\theta p}}{1-\left(\kappa_Y^{m-2}\sum_{i=2}^{m}|L_i|.|s_i(t)|^p\right)^{\theta}}<\infty$$

for all $x\in X\setminus\{0\}$. This proves that (1.4) is satisfied.

According to Corollary 1.4, with a large positive integer t, there exists a fixed point $f_t : X\to Y$ of $T_t f_t(x)=f_t(x)$ satisfying

$$\|f_t(x)-f(x)\|_Y^{\theta}$$
$$\le 4\varepsilon_t^*(x)$$
$$=\frac{4\alpha_t^{\theta}\|x\|_X^{\theta p}}{1-\left(\kappa_Y^{m-2}\sum_{i=2}^{m}|L_i|.|s_i(t)|^p\right)^{\theta}} \qquad (2.12)$$

for all $x\in X\setminus\{0\}$. Furthermore, by (1.6) we obtain

$$f_t(x)=\lim_{n\to\infty}T_t^n f(x). \qquad (2.13)$$

By induction, we will show that for all $x\in X\setminus\{0\}$, $r\in\mathbb{N}$

$$\left\|\sum_{i=1}^{m}L_iT_t^r f\left(\sum_{j=1}^{n}a_{ij}x_j\right)\right\|_Y$$
$$\le\left(\kappa_Y^{m-2}\sum_{i=2}^{m}|L_i|.|s_i(t)|^p\right)^{r}C\sum_{j=1}^{n}\|x_j\|_X^p. \qquad (2.14)$$

Indeed, if $r=0$, then (2.14) holds by (2.1). Suppose that (2.14) holds for $r=l$, that is,

$$\left\|\sum_{i=1}^{m}L_iT_t^l f\left(\sum_{j=1}^{n}a_{ij}x_j\right)\right\|_Y$$
$$\le\left(\kappa_Y^{m-2}\sum_{i=2}^{m}|L_i|.|s_i(t)|^p\right)^{l}C\sum_{j=1}^{n}\|x_j\|_X^p.$$

We have

$$\left\|\sum_{i=1}^{m}L_iT_t^{l+1} f(\sum_{j=1}^{n}a_{ij}x_j)\right\|_Y$$
$$=\left\|\sum_{i=1}^{m}L_i\sum_{k=2}^{m}L_kT_t^l f\Big(s_k(t)\sum_{j=1}^{n}a_{ij}x_j\Big)\right\|_Y$$
$$=\left\|\sum_{k=2}^{m}L_k\sum_{i=1}^{m}L_iT_t^l f\Big(\sum_{j=1}^{n}a_{ij}s_k(t)x_j\Big)\right\|_Y$$
$$\le\kappa_Y^{m-2}\sum_{k=2}^{m}|L_k|\left\|\sum_{i=1}^{m}L_iT_t^l f\Big(\sum_{j=1}^{n}a_{ij}s_k(t)x_j\Big)\right\|_Y$$
$$\le\kappa_Y^{m-2}\sum_{k=2}^{m}|L_k|\left(\kappa_Y^{m-2}\sum_{i=2}^{m}|L_i|.|s_i(t)|^p\right)^{l}\sum_{j=1}^{n}C\|s_k(t)x_j\|_X^p$$
$$=\left(\kappa_Y^{m-2}\sum_{k=2}^{m}|L_k|.|s_k(t)|^p\right).\left(\kappa_Y^{m-2}\sum_{i=2}^{m}|L_i|.|s_i(t)|^p\right)^{l}$$
$$.C\sum_{j=1}^{n}\|x_j\|_X^p$$
$$=\left(\kappa_Y^{m-2}\sum_{i=2}^{m}|L_i|.|s_i(t)|^p\right)^{l+1}\sum_{j=1}^{n}C\|x_j\|_X^p.$$

So, (2.14) holds for all $r\in\mathbb{N}$.

By using (2.10), we gain

$$\lim_{r\to\infty}\left(\kappa_Y^{m-2}\sum_{i=2}^{m}|L_i|.|s_i(t)|^p\right)^{r}C\sum_{j=1}^{n}\|x_j\|_X^p=0. \qquad (2.15)$$

From (2.13), (2.14), (2.15), Remark 1.2 (2) and Theorem 1.5, we obtain

$$\left\|\sum_{i=1}^{m}L_i f_t\left(\sum_{j=1}^{n}a_{ij}x_j\right)\right\|_Y$$
$$=\left\|\lim_{r\to\infty}\sum_{i=1}^{m}L_iT_t^{r+1} f\left(\sum_{j=1}^{n}a_{ij}x_j\right)\right\|_Y$$

$$= \lim_{r\to\infty} \left\| \left\| \sum_{i=1}^{m} L_i T_t^{r+1} f\left(\sum_{j=1}^{n} a_{ij} x_j\right) \right\| \right\|_Y$$

$$\le \lim_{r\to\infty} \left\| \sum_{i=1}^{m} L_i T_t^{r+1} f\left(\sum_{j=1}^{n} a_{ij} x_j\right) \right\|_Y$$

$$\le \lim_{r\to\infty} \left(\kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . |s_i(t)|^p \right)^r C \sum_{j=1}^{n} \| x_j \|_X^p$$

$$= 0.$$

It means that

$$\sum_{i=1}^{m} L_i f_t \left(\sum_{j=1}^{n} a_{ij} x_j\right) = 0. \qquad (2.16)$$

So, f_t satisfies (2.2) with a large positive integer t. Letting $t \to \infty$ in (2.12) and using (2.5), we gain

$$\lim_{t\to\infty} \| f_t(x) - f(x) \|_Y^{\theta}$$

$$\le \lim_{t\to\infty} \frac{4\alpha_t^{\theta} \| x \|_X^{\theta p}}{1 - \left(\kappa_Y^{m-2} \sum_{i=2}^{m} |L_i| . |s_i(t)|^p \right)^{\theta}}$$

$$= 0.$$

It follows that $\lim_{t\to\infty} \| f_t(x) - f(x) \|_Y = 0.$ So

$$\lim_{t\to\infty} f_t(x) = f(x). \qquad (2.17)$$

Letting $t \to \infty$ in (2.16) and using (2.17), we gain $\sum_{i=1}^{m} L_i f\left(\sum_{j=1}^{n} a_{ij} x_j\right) = 0$ for all $x_1, \ldots, x_n \in X \setminus \{0\}$. So, f satisfies (2.2). □

We continue to present an extension of (Zhang, 2015, Theorem 1.7) from normed spaces to quasi-normed spaces.

Theorem 2.2. *Suppose that*

1. $\mathbb{F}, \mathbb{K}$ *denote the fields of real or complex numbers and* $(X, \|.\|_X, \kappa_X)$ *is a quasi-normed space over field* $\mathbb{F}$, $(Y, \|.\|_Y, \kappa_Y)$ *is a quasi-Banach space over field* $\mathbb{K}$ *and* $f: X \to Y$ *is a given mapping.*

2. $n \ge 2$ *and* m *are positive integers,* $C \ge 0,$ $a_{ij} \in \mathbb{F}$ *and* $L_i \in \mathbb{K}$ *are given parameters for* $i = 1, \ldots, m,$ $j = 1, \ldots, n.$

3. *There exist* $i_0 \in \{1, \ldots, m\}$ *and* $j_1 \ne j_2 \in \{1, \ldots, n\}$ *such that* $a_{i_0 j_1} \ne 0,$ $a_{i_0 j_2} \ne 0.$ *For all* $i \ne i_0,$ $\gamma \ne 0,$ *there is* $j \in \{1, \ldots, n\}$ *satisfying* $a_{ij} \ne \gamma a_{i_0 j}.$

4. *There exists* $p_1, \ldots, p_n \in \mathbb{R}$ *such that* $p_1 + \cdots + p_n < 0$ *and*

$$\left\| \sum_{i=1}^{m} L_i f\left(\sum_{j=1}^{n} a_{ij} x_j\right) \right\|_Y \le C \prod_{j=1}^{n} \| x_j \|_X^{p_j}$$

for all $x_1, \ldots, x_n \in X \setminus \{0\}.$

Then we have

$$\sum_{i=1}^{m} L_i f\left(\sum_{j=1}^{n} a_{ij} x_j\right) = 0$$

for all $x_1, \ldots, x_n \in X \setminus \{0\}.$

Proof. Set $\alpha_t := C \prod_{j=1}^{n} |k_j t + b_j|^{p_j}$. Then $\lim_{t\to\infty} \alpha_t = 0$ since $\sum_{j=1}^{n} p_j < 0.$ The proof of Theorem 2.2 is now the same as the that of Theorem 2.1. □

We apply the established result to prove some results of Zhang (2015).

Corollary 2.3 (Zhang, 2015, Theorem 1.6)**.** *Suppose that*

1. $\mathbb{F}, \mathbb{K}$ *denote the fields of real or complex numbers and* $(X, \|.\|_X)$ *is a normed space over field* $\mathbb{F}$, $(Y, \|.\|_Y)$ *is a Banach space over field* $\mathbb{K}$ *and* $f: X \to Y$ *is a given mapping.*

2. $n \ge 2$ *and* m *are positive integers,* $C \ge 0,$ $a_{ij} \in \mathbb{F}$ *and* $L_i \in \mathbb{K}$ *are given parameters for* $i = 1, \ldots, m,$ $j = 1, \ldots, n.$

3. *There exist* $i_0 \in \{1,\ldots,m\}$ *and* $j_1 \neq j_2 \in \{1,\ldots,n\}$ *such that* $a_{i_0 j_1} \neq 0$, $a_{i_0 j_2} \neq 0$. *For all* $i \neq i_0$, $\gamma \neq 0$, *there is* $j \in \{1,\ldots,n\}$ *satisfying* $a_{ij} \neq \gamma a_{i_0 j}$.

4. *There exists* $p < 0$ *such that*

$$\left\| \sum_{i=1}^{m} L_i f\left(\sum_{j=1}^{n} a_{ij} x_j\right) \right\|_Y \leq C \sum_{j=1}^{n} \| x_j \|_X^p$$

for all $x_1,\ldots,x_n \in X \setminus \{0\}$.

Then

$$\sum_{i=1}^{m} L_i f\left(\sum_{j=1}^{n} a_{ij} x_j\right) = 0$$

for all $x_1,\ldots,x_n \in X \setminus \{0\}$.

Proof. The normed spaces are the quasi-normed spaces when $\kappa = 1$. So, all assumptions of Theorem 2.1 are satisfied. Then Corollary 2.3 follows from Theorem 2.1. □

We continue to apply established results to some special cases. The next is an extension of the result of Zhang (2015) from the normed spaces to the quasi-normed spaces.

Proposition 2.4. *Let* $\mathbb{F}, \mathbb{K}$ *denote the fields of real or complex numbers,* $(X, \|.\|_X, \kappa_X)$ *is a quasi-normed space over field* $\mathbb{F}$, $(Y, \|.\|_Y, \kappa_Y)$ *is a quasi-Banach space* $\mathbb{K}$, $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, $c \geq 0$, $p < 0$ *and let* $f : X \to Y$ *satisfy*

$$\| f(ax+by) - Af(x) - Bf(y) \|_Y$$
$$\leq c(\| x \|_X^p + \| y \|_X^p)$$

for all $x, y \in X \setminus \{0\}$. *Then* f *satisfies the equation*

$$f(ax+by) - Af(x) - Bf(y) = 0$$

for all $x, y \in X \setminus \{0\}$.

Proof. We set $A_1 := (a,b)$, $A_2 := (1,0)$ and $A_3 := (0,1)$. For all $i \in \{2,3\}$ and $\gamma \in \mathbb{F}$, we gain $A_i \neq \gamma A_1$. Furthermore, f satisfies (2.1) for all $x, y \in X \setminus \{0\}$, $L_1 = 1$, $L_2 = -A$ and $L_3 = -B$. So all assumptions of Theorem 2.1 are satisfied. Then Proposition 2.4 follows from Theorem 2.1. □

Proposition 2.5. *Let* $\mathbb{F}, \mathbb{K}$ *denote the fields of real or complex numbers,* $(X, \|.\|_X, \kappa_X)$ *is a quasi-normed space over field* $\mathbb{F}$, $(Y, \|.\|_Y, \kappa_Y)$ *is a quasi-Banach space* $\mathbb{K}$, $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, $c \geq 0$, $p < 0$ *and let* $f : X \to Y$ *satisfy*

$$\left\| \sum_{i=1}^{n} (-1)^{n-i} C_n^i f(ix+y) - n!\, f(x) \right\|_Y$$
$$\leq c(\| x \|_X^p + \| y \|_X^p)$$

for all $x, y \in X \setminus \{0\}$. *Then* f *satisfies the equation*

$$\sum_{i=1}^{n} (-1)^{n-i} C_n^i f(ix+y) - n!\, f(x) = 0$$

for all $x, y \in X \setminus \{0\}$.

Proof. We set $A_1 := (1;1)$, $A_i := (i,1)$ for all $i \in \{2,\ldots,n\}$ and $A_{n+1} := (1,0)$. For all $i \in \{2, \text{n+1}\}$ and $\gamma \in \mathbb{F}$, we gain $A_i \neq \gamma A_1$. Furthermore, f satisfies (2.1) for all $x, y \in X \setminus \{0\}$ and $L_1, L_2, \ldots, L_n$ are $(-1)^n C_n^0$, $(-1)^{n-1} C_n^1, \ldots, 1$, respectively, and $L_{n+1} = -n!$. So all assumptions of Theorem 2.1 are satisfied. Then Proposition 2.5 follows from Theorem 2.1. □

Proposition 2.6. *Let* $\mathbb{F}, \mathbb{K}$ *denote the fields of real or complex numbers,* $(X, \|.\|_X, \kappa_X)$ *is a quasi-normed space over field* $\mathbb{F}$, $(Y, \|.\|_Y, \kappa_Y)$ *is a quasi-Banach space* $\mathbb{K}$, $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, $c \geq 0$, $p < 0$ *and let* $f : X \to Y$ *satisfy*

$$\| f(x+y) + f(y+z) + f(x+z)$$
$$- f(x) - f(y) - f(z) - f(x+y+z) \|_Y$$
$$\leq c(\| x \|_X^p + \| y \|_X^p + \| z \|_X^p)$$

for all $x, y, z \in X \setminus \{0\}$. *Then* f *satisfies the equation*

$$f(x+y)+f(y+z)+f(x+z)$$
$$= f(x)+f(y)+f(z)+f(x+y+z)$$

for all $x, y, z \in X \setminus \{0\}$.

Proof. We set $A_1 := (1,1,0)$, $A_2 := (0,1,1)$, $A_3 := (1,0,1)$, $A_4 := (1,0,0)$, $A_5 := (0,1,0)$, $A_6 := (0,0,1)$ and $A_7 := (1,1,1)$. For all $i \in \{2,3,4,5,6,7\}$ and $\gamma \in \mathbb{F}$, we gain $A_i \neq \gamma A_1$. Furthermore, f satisfies (2.1) for all $x, y, z \in X \setminus \{0\}$ and $L_1 = L_2 = L_3 = 1$, $L_4 = L_5 = L_6 = L_7 = -1$. So all assumptions of Theorem 2.1 are satisfied. Then Proposition 2.6 follows from Theorem 2.1. □

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