ANNIHILATOR OF LOCAL COHOMOLOGY MODULES AND STRUCTURE OF RINGS

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ABSTRACT

Let (*R*, m) be a Noetherian local ring, *A* an Artinian *R*-module, and *M* a finitely generated *R*-module. It is clear that Ann *R*(*M/* p *M*) = p, for all p ∈ Var(Ann *R M*). Therefore, it is natural to consider the following dual property for annihilator of Artinian modules:

Ann *R*(0 : *A* p) = p, for all p ∈ Var(Ann *R A*). (∗)

Let *i* ≥ 0 be an integer. Alexander Grothendieck showed that the local cohomology module *H*m*i* (*M*) of *M* is Artinian. The property (∗) of local cohomology modules is closed related to the structure of the base ring. In this paper, we prove that for each p ∈ Spec(*R*) such that *H*m*i* (*R/* p) satisfies the property (*) for all *i*, then *R/* p is universally catenary and the formal fibre of *R* over p is Cohen-Macaulay.

Keywords: *Local cohomology; universally catenary; formal fibre; Artinian module; CohenMacaulay ring*

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LINH HÓA TỬ CỦA MÔĐUN ĐỐI ĐỒNG ĐIỀU ĐỊA PHƯƠNG VÀ CẤU TRÚC VÀNH

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TÓM TẮT

Cho (*R*, m) là vành Noether địa phương, *A* là *R*-môđun Artin, và *M* là *R*-môđun hữu hạn sinh. Ta có Ann *R*(*M/* p *M*) = p với mọi p ∈ Var(Ann *R M*). Do đó rất tự nhiên ta xét tính chất sau về linh hóa tử của môđun Artin

Ann *R*(0 : *A* p) = p for all p ∈ Var(Ann *R A*). (∗)

Cho *i* ≥ 0 là số nguyên. Alexander Grothendieck đã chỉ ra rằng môđun đối đồng điều địa phương *Hi* m(*M*) là Artin. Tính chất (∗) của các môđun đối đồng điều địa phương liên hệ mật thiết với cấu trúc vành cơ sở. Trong bài báo này, chúng tôi chỉ ra với mỗi p ∈ Spec(*R*) mà *H*m*i* (*R/* p) thỏa mãn tính chất (*) với mọi *i* thì *R/* p là catenary phổ dụng và các thớ hình thức của *R* trên p là Cohen-Macaulay.

Từ khóa: *Đối đồng điều địa phương; catenary phổ dụng; thớ hình thức; môđun Artin; vành Cohen-Macaulay*

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1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring, A an Artinian R-module, and M a finitely generated R-module of dimension d. For each ideal I of R, we denote by $\mathrm{Var}(I)$ the set of all prime ideals containing I. For a subset T of $\mathrm{Spec}(R)$, we denote by $\min(T)$ the set of all minimal elements of T under the inclusion.

It is clear that $\mathrm{Ann}_R(M/\mathfrak{p}M) = \mathfrak{p}$, for all $\mathfrak{p} \in \mathrm{Var}(\mathrm{Ann}_R M)$. Therefore, it is natural to consider the following dual property for annihilator of Artinian modules:

$$Ann_R(0 :_A \mathfrak{p}) = \mathfrak{p}, \forall \mathfrak{p} \in \mathrm{Var}(\mathrm{Ann}_R A).(*)$$

If R is complete with respect to $\mathfrak{m}$-adic topology, it follows by Matlis duality that the property (*) is satisfied for all Artinian R-modules. However, there are Artinian modules which do not satisfy this property. For example, by [1, Example 4.4], the Artinian R-module $H^1_{\mathfrak{m}}(R)$ does not satisfy the property (*), where R is the Noetherian local domain of dimension 2 constructed by M. Ferrand and D. Raynaud [2] (see also [3, App. Ex. 2] Ex. 2]) such that its $\mathfrak{m}$-adic completion $\widehat{R}$ has an associated prime $\mathfrak{q}$ of dimension 1. In [4], N. T. Cuong, L. T. Nhan and N. T. Dung showed that the top local cohomology module $H^d_{\mathfrak{m}}(M)$ satisfies property $(*)$ if and only if the ring $R/\mathrm{Ann}_R(M/U_M(0))$ is catenary, where $U_M(0)$ is the largest submodule of M of dimension less than d. The property $(*)$ of local cohomology modules is closed related to the structure of the ring. In [5], L. T. Nhan and the author proved that if $H^i_{\mathfrak{m}}(M)$ satisfies the property (*) for all i , then $R/\mathfrak{p}$ is unmixed for all $\mathfrak{p} \in \mathrm{Ass}\, M$ and the ring $R/\mathrm{Ann}_R M$ is universally catenary. The following conjecture was given by N. T. Cuong in his seminar.

Conjecture 1.1. *The following statements are equivalent:*

(i) $H^i_{\mathfrak{m}}(R)$ satisfies the property (*) for all i;

(ii) R is universally catenary and all its formal flbers are Cohen-Macaulay.

L. T. Nhan and T. D. M. Chau proved in [6] that $H^i_{\mathfrak{m}}(M)$ satisfies the property (*) for all i, for all finitely generated R-module M if and only if R is universally catenary and all its formal flbers are Cohen-Macaulay. The following result is the main result of this paper. We hope that we can use this to give a positive answer for the above conjecture.

Theorem 1.2. *Assume $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $H^i_{\mathfrak{m}}(R/\mathfrak{p})$ satisfies the property (*) for all i. Then $R/\mathfrak{p}$ is universally catenary and the formal fibre of R over $\mathfrak{p}$ is Cohen-Macaulay.*

2. Proof of the main results

The theory of secondary representation was introduced by I. G. Macdonald (see [7]) which is in some sense dual to that of primary decomposition for Noetherian modules. Note that every Artinian R-module A has a minimal secondary representation $A = A_1 + \ldots + A_n$, where A_i is $\mathfrak{p}_i$-secondary. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the choice of the minimal secondary representation of A. This set is called *the set of attached prime ideals* of A, and denoted by $\mathrm{Att}_R A$. Note also that A has a natural structure as an $\widehat{R}$-module. With this structure, a subset of A is an R-submodule if and only if it is an $\widehat{R}$-submodule of A. Therefore, A is an Artinian $\widehat{R}$-module.

Lemma 2.1. *(i) The set of all minimal elements of* $\mathrm{Att}_R A$ *is exactly the set of all minimal elements of* $\mathrm{Var}(\mathrm{Ann}_R A)$.

(ii) $\mathrm{Att}_R A = \{\widehat{\mathfrak{p}} \cap R \ : \ \widehat{\mathfrak{p}} \in \mathrm{Att}_{\widehat{R}} A\}$.

R. N. Roberts introduced the concept of Krull dimension for Artinian modules (see [8]). D. Kirby changed the terminology of Roberts and referred to Noetherian dimension to avoid confusion with Krull dimension defined for finitely generated modules (see [9]). The Noetherian dimension of A is denoted by $\text{N-dim}_R(A)$. In this paper, we use the terminology of Kirby (see [9]).

Lemma 2.2 ([1]). *(i)* $\text{N-dim}_R(A) \leqslant \dim(R/\operatorname{Ann}_R A)$, *and the equality holds if A satisfies the property (*).*

(ii) $\text{N-dim}_R(H^i_{\mathfrak{m}}(M)) \le i$, *for all i.*

The following property of attached primes of the local cohomology under localization is known as Weak general Shifted Localization Principle (see [10]).

Lemma 2.3. *We have* $\operatorname{Att}_{R_\mathfrak{p}}(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p} R_\mathfrak{p}}(M_\mathfrak{p}))$ *is the subset of* $\{\mathfrak{q} R_\mathfrak{p} \mid \mathfrak{q} \in \min \operatorname{Att}_R(H^i_{\mathfrak{m}}(M)), \mathfrak{q} \subseteq \mathfrak{p}\}$, *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$.

For an integer $i \ge 0$, following M. Brodmann and R. Y. Sharp (see [11]), the i-th pseudo support of M, denoted by $\operatorname{Psupp}^i_R(M)$, is defined by the set

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p} R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0\}.$$

Note that the role of $\operatorname{Psupp}^i_R(M)$ for the Artinian R-module $A = H^i_{\mathfrak{m}}(M)$ is in some sense similar to that of $\operatorname{Supp} L$ for a finitely generated R-module L, cf. [11], [5]. Although, we always have $\operatorname{Supp} L = \operatorname{Var}(\operatorname{Ann}_R L)$, but the analogous equality $\operatorname{Psupp}^i_R(M) = \operatorname{Var}(\operatorname{Ann}_R H^i_{\mathfrak{m}}(M))$ is not valid in general. The following lemma gives a necessary and sufficient conditions for the above equality.

Lemma 2.4 ([5]). *Let $i \ge 0$ be an integer. Then the following statements are equivalent:*

(i) $H^i_{\mathfrak{m}}(M)$ *satisfies the property (*).*

(ii) $\operatorname{Var}\left(\operatorname{Ann}_R(H^i_{\mathfrak{m}}(M))\right) = \operatorname{Psupp}^i_R M$.

In particular, if $H^i_{\mathfrak{m}}(M)$ satisfies the property (*) then

$$\min \operatorname{Att}_R(H^i_{\mathfrak{m}}(M)) = \min \operatorname{Psupp}^i_R M.$$

In 2010, N. T. Cuong, L. T. Nhan and N. T. K. Nga (see [12]) used pseudo support to describe the non-Cohen-Macaulay locus of M. Recall that M is *equidimensional* if $\dim(R/\mathfrak{p}) = d$, for all $\mathfrak{p} \in \min(\operatorname{Ass} M)$.

Lemma 2.5 ([12]). *Suppose that M is equidimensional and the ring $R/\operatorname{Ann}_R M$ is catenary. Then $\operatorname{Psupp}^i_R(M)$ is closed for $i = 0, 1, d$ and* $\operatorname{nCM}(M) = \bigcup_{i=0}^{d-1} \operatorname{Psupp}^i_R(M)$, *where* $\operatorname{nCM}(M)$ *is the Non Cohen-Macaulay locus of M.*

Following M. Nagata ([3]), we say that M is *unmixed* if $\dim(\widehat{R}/\widehat{\mathfrak{p}}) = d$ for all prime ideals $\widehat{\mathfrak{p}} \in \operatorname{Ass} \widehat{M}$, and M is *quasi unmixed* if $\widehat{M}$ is equidimensional. The next lemma show that the property (*) for the local cohomology modules $H^i_{\mathfrak{m}}(M)$ of levels $i < d$ is closed related to the universal catenaricity and unmixedness of certain local rings.

Lemma 2.6 ([5]). *Assume that $H^i_{\mathfrak{m}}(M)$ satisfies the property (*) for all $i < d$. Then $R/\mathfrak{p}$ is unmixed for all $\mathfrak{p} \in \operatorname{Ass} M$ and the ring $R/\operatorname{Ann}_R M$ is universally catenary.*

Proof of Theorem 1.2. It follows from the Lemma 2.6 that $R/\mathfrak{p} = R/\operatorname{Ann}_R(R/\mathfrak{p})$ is universally catenary.

Set S to be the image of $R \setminus \mathfrak{p}$ in $\widehat{R}$. We have

$$R_\mathfrak{p}/\mathfrak{p} R_\mathfrak{p} \otimes_R \widehat{R} \cong S^{-1}(\widehat{R}/\mathfrak{p} \widehat{R}).$$

We need to prove $(S^{-1}(\widehat{R}/\mathfrak{p} \widehat{R}))_{S^{-1}\widehat{\mathfrak{q}}}$ is Cohen-Macaulay for all $\widehat{\mathfrak{q}} \in \operatorname{Spec}(\widehat{R})$ such that $(\widehat{\mathfrak{q}} \cap R) \cap S = \emptyset$. Assume that the statement is not true. Since

$$(S^{-1}(\widehat{R}/\mathfrak{p}\widehat{R}))_{S^{-1}\widehat{\mathfrak{q}}} \cong (\widehat{R}/\mathfrak{p}\widehat{R})_{\widehat{\mathfrak{q}}}$$

as $\widehat{R}_{\widehat{\mathfrak{q}}}$-module, there exists $\widehat{\mathfrak{q}} \in \operatorname{Spec}(\widehat{R}), \widehat{\mathfrak{q}} \cap S = \emptyset$ such that $(\widehat{R}/\mathfrak{p}\widehat{R})_{\widehat{\mathfrak{q}}}$ is not Cohen-Macaulay. Then there exists $\widehat{\mathfrak{p}} \in \operatorname{Spec}(R), \widehat{\mathfrak{q}} \supseteq \widehat{\mathfrak{p}}$, $(\widehat{\mathfrak{p}} \cap R) \cap S = \emptyset$ and $\widehat{\mathfrak{p}} \in \operatorname{Min}\operatorname{nCM}(\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})$. Hence,

$$\operatorname{nCM}((\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})_{\widehat{\mathfrak{p}}}) = \left\{\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}}\right\}.$$

We have $R/\mathfrak{p}$ is unmixed by Lemma 2.6. So $\widehat{R}/\widehat{\mathfrak{p}}\widehat{R}$ is equidimensional. Hence $(\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})_{\widehat{\mathfrak{p}}}$ is equidimensional. On the other hand, since $(\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})_{\widehat{\mathfrak{p}}}$ is the image of a Cohen-Macaulay ring, $(\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})_{\widehat{\mathfrak{p}}}$ is generalized Cohen-Macaulay.

Set $s = \dim \widehat{R}/\widehat{\mathfrak{p}}\widehat{R} = \operatorname{ht}(\widehat{\mathfrak{p}}/\mathfrak{p}\widehat{R})$. By Lemma 2.5, we have

$$\operatorname{nCM}(\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})_{\widehat{\mathfrak{p}}} = \bigcup_{i=0}^{s-1} \operatorname{Psupp}^i_{\widehat{R}}((\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})_{\widehat{\mathfrak{p}}}).$$

Therefore, there exists $i < s$ such that $H^i_{\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}}}(\widehat{R}/\mathfrak{p}\widehat{R})_{\widehat{p}} \neq 0$. On the other hand,

$$\ell(H^i_{\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}}}(\widehat{R}/\mathfrak{p}\widehat{R})_{\widehat{p}}) < \infty.$$

Then

$$\operatorname{Att}_{\widehat{R}}(H^i_{\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}}}(\widehat{R}/\mathfrak{p}\widehat{R})_{\widehat{p}}) = \left\{\mathfrak{p}\widehat{R}_{\widehat{\mathfrak{p}}}\right\}.$$

It is followed by Weak general Shifted Localization Principle (Lemma 2.3) that $\widehat{\mathfrak{p}} \in \operatorname{Att}_{\widehat{R}}(H^{i+\dim \widehat{R}/\widehat{\mathfrak{p}}}_{\mathfrak{m}}(\widehat{R}/\mathfrak{p}\widehat{R}))$. Set $j = i + \dim \widehat{R}/\widehat{\mathfrak{p}}$. We have

$$\begin{aligned} j < \operatorname{ht}\widehat{\mathfrak{p}}/\mathfrak{p}\widehat{R} + \dim \widehat{R}/\widehat{\mathfrak{p}} &\leq \dim \widehat{R}/\mathfrak{p}\widehat{R} \\ &= \dim R/\mathfrak{p}. \end{aligned}$$

Hence, $\mathfrak{p} \in \operatorname{Att}_R(H^j_{\mathfrak{m}}(R/\mathfrak{p}))$ by Lemma 2.1. By Lemma 2.2

$$\begin{aligned} \text{N-dim}\, H^j_{\mathfrak{m}}(R/\mathfrak{p}) \leq j < \dim R/\mathfrak{p} \\ \leq R/\operatorname{Ann}_R H^j_{\mathfrak{m}}(R/\mathfrak{p}). \end{aligned}$$

This implies that $H^j_{\mathfrak{m}}(R/\mathfrak{p})$ does not satisfy the property (*). It is in contradiction to the hypothesis. Therefore, all its formal fibers over $\mathfrak{p}$ are Cohen-Macaulay. □

3. Conclusion

The paper gives a relation between the property (*) of local cohomology module and structure of base ring. In detail, we prove that for each $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $H^i_{\mathfrak{m}}(R/\mathfrak{p})$ satisfies the property (*) for all i, then $R/\mathfrak{p}$ is universally catenary and the formal fibre of R over $\mathfrak{p}$ is Cohen-Macaulay.

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