ON UNIQUENESS OF MEROMORPHIC FUNCTIONS PARTIALLY SHARING VALUES WITH THEIR SHIFTS

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ABSTRACT

In 1926, R. Nevanlinna showed that two distinct nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$ share five distinct values then $f = g$ on whole $\mathbb{C}$. If a meromorpic function f with hyper-order less than 1 and its shifts g share four distinct values or share partially four small periodic functions in the complex plane, then whether $f = g$ or not. Our aim is to study uniqueness of such meromorphic functions. For our purpose, we use techniques in Nevanlinna theory by estimating the counting functions and use the property of defect relation of values on the complex plane. Let a_1, a_2, a_3, a_4 be four small periodic functions with period c in the complex plane for $c \in \mathbb{C} \setminus \{0\}$. Then we prove a result as folows: Assume that meromorphic function f of hyper-order less than 1 with its shift $f(z+c)$ share a_3 CM, shares partially a_1, a_2 IM and reduced defect of f at a_4 is maximal. Then under an appropriate deficiency assumption, $f(z) = f(z+c)$ for all $z \in \mathbb{C}$. Our result is a continuation of previous works of the authors and provides an understanding of the meromorphic functions of hyper-order less than 1.

Keywords: *meromorphic function; sharing partially values; uniqueness theorem; periodic function; deficiency*

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VỀ TÍNH DUY NHẤT CỦA CÁC HÀM PHÂN HÌNH CHIA SẺ MỘT PHẦN CÁC GIÁ TRỊ CÙNG VỚI CÁC HÀM DỊCH CHUYỂN CỦA CHÚNG

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TÓM TẮT

Năm 1926, R. Nevanlinna chỉ ra rằng hai hàm phân hình khác hằng f và g trên mặt phẳng phức $\mathbb{C}$ chia sẻ năm giá trị khác nhau IM thì $f = g$ trên toàn bộ $\mathbb{C}$. Nếu một hàm phân hình $f(z)$ có siêu bậc nhỏ hơn 1 và hàm dịch chuyển $f(z+c)$ của nó chia sẻ bốn giá trị phân biệt hoặc chia sẻ bốn hàm nhỏ tuần hoàn trong mặt phẳng phức, thì liệu $f(z) = f(z+c)$ với mọi $z \in \mathbb{C}$ hay không? Mục đích của chúng tôi là nghiên cứu tính duy nhất của những hàm phân hình trong tình huống như thế. Để đạt được mục đích, chúng tôi sử dụng kĩ thuật trong lí thuyết Nevanlinna bằng cách dựa vào ước lượng các hàm đếm và sử dụng tích chất của tổng số khuyết của các giá trị trong mặt phẳng phức. Xét bốn hàm nhỏ a_1, a_2, a_3, a_4 tuần hoàn với chu kì c trong mặt phẳng phức với $c \in \mathbb{C} \setminus \{0\}$. Chúng tôi chứng minh được kết quả như sau: Giả sử rằng hàm phân hình $f(z)$ có siêu bậc nhỏ hơn 1 cùng với hàm dịch chuyển của nó $f(z+c)$ chia sẻ a_3 CM, chia sẻ một phần a_1, a_2, đồng thời số khuyết thu gọn của f tại a_4 là cực đại. Thế thì dưới điều kiện về số khuyết tại một giá trị bất kì khác a_4, ta có $f(z) = f(z+c)$ với mọi $z \in \mathbb{C}$. Kết quả của chúng tôi là sự tiếp tục các công việc trước đó của các tác giả và nó cung cấp cho chúng ta có thêm hiểu biết về những hàm phân hình có siêu bậc nhỏ hơn 1.

Từ khóa: *Hàm phân hình; chia sẻ một phần các giá trị; định lí duy nhất; hàm tuần hoàn; số khuyết*

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1. Introduction

In this article, we consider meromorphic functions in the whole complex plane $\mathbb{C}$. We denote proximity function and Nevanlinna characteristic function of f by $m(r,f)$ and $T(r,f)$ respectively. For each a meromorphic function a in the extended complex plane, we denote by $N(r,\frac{1}{f-a})$ the zeros counting function of $f-a$ with counting multiplicities and $\overline{N}(r,\frac{1}{f-a})$ the zeros counting function of $f-a$ without counting multiplicities. We use sympol $N(r,f)$ instead of notation $N(r,\frac{1}{f-\infty})$ and $\overline{N}(r,f)$ instead of $\overline{N}(r,\frac{1}{f-\infty})$. The deficiency and reduced deficiency of a with respect to f are defined respectively by

$$\delta(a,f)=1-\limsup_{r\to\infty}\frac{N(r,\frac{1}{f-a})}{T(r,f)},\ \Theta(a,f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}.$$

The hyper-order $\gamma(f)$ of a meromorphic function f are defined by

$$\gamma(f)=\limsup_{r\to\infty}\frac{\log^+(\log^+T(r,f))}{\log r}.$$

Denote by $S(r,f)$ a quantity equal to $o(T(r,f))$ for all $r\in(1,\infty)$ outside a finite Borel measure set. In particular, we denote by $S_1(r,f)$ any quantity satisfying $S_1(r,f)=o(T(r,f))$ as $r\to\infty$ outside of a possible exceptional set of finite logarithmic measure.

Let f and g be two meromorphic functions and a function meromorphic a. We say that f and g share a IM when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share a CM.

For positive integers k (may be $k=+\infty$), we denote by $\overline{E}_{k)}(a,f)$ the set of zeros of $f-a$ with multiplicity $l\le k$, where a zero with multiplicity l is counted only once in the set. The reduced counting function corresponding to $\overline{E}_{k)}(a,f)$ is denoted by $\overline{N}_{k)}(r,\frac{1}{f-a})$. Similarly, we also denote by $\overline{N}_{(k}(r,\frac{1}{f-a})$ the reduced counting function of those a-points of f whose multiplicities are not less than k in counting the a-points of f. If $k=+\infty$, we omit character k in the notation.

Uniqueness questions of meromorphic functions and their shifts sharing values have been treated as well [1]-[6]. In particular, in 2016 K. S. Charak, R. J. Korhonen and G. Kumar [7] gave a result of partially shared values and obtained the following theorem under an appropriate deficiency assumption.

Theorem A [7]: *Let f be a nonconstant meromorphic function of hyper-order $\gamma(f)<1$ and $c\in\mathbb{C}\setminus\{0\}$. Let $a_1,a_2,a_3,a_4\in\hat{S}(f)$ be four distinct periodic functions with period c. If $\delta(a,f)>0$ for some $a\in\hat{S}(f)$ and*

$$\overline{E}(a_j,f(z))\subseteq\overline{E}(a_j,f(z+c)),\ j=1,2,3,4$$

then $f(z)=f(z+c)$ for all $z\in\mathbb{C}$.

Here, we denote $S(f)$ as the family of all small functions of f and $\hat{S}(f):=S(f)\cup\{\infty\}$.

Recently, W. Lin, X. Lin and A. Wu [8] obtained a counterexample which showed that Theorem A does not hold when the condition "partially shared values $\overline{E}(a_j,f(z))\subseteq\overline{E}(a_j,f(z+c)),\ j=1,2$" is replaced by the condition "truncated partially shared values $\overline{E}_{k)}(a_j,f(c))\subseteq\overline{E}_{k)}(a_j,f(z+c)),\ j=1,2$", even if $f(z)$ and $f(z+c)$ share a_3,a_4 CM as follows:

Example B [8]: *Let* $f(z)=\sin z$ *and* $c=\pi$. *It is easy to see that* $f(z)$ *have hyper-oder* $\gamma(f)<1$ *and shares* 0 *and* ∞ *CM with its shift* $f(z+c)$ *and* $\overline{E}_{1)}(\pm 1, f(z))=\overline{E}_{1)}(\pm 1, f(z+c))=\varnothing,$ *but* $f(z+c)=-f(z)$ *for all* $z\in\mathbb{C}$. *Althought, the condition* $\delta(\infty, f)=\Theta(\infty, f)=1>0$ *is satisfied.*

A question is arised naturally at this moment: If $\delta(\infty, f)>0$ for some $a\neq\infty$ then wheather we obtain an uniqueness theorem in the situation of Example B.

Our aim in this paper is to give positive answer for this question. Namely, we have prove the following.

Theorem: *Let* f *be a nonconstant meromorphic function of hyper-order* $\gamma(f)<1$ *and* $c\in\mathbb{C}\setminus\{0\}$. *Let* $a_1,a_2,a_3,a_4\in\hat{S}(f)$ *be four distinct periodic functions with period* c *such that* $\Theta(a_4, f)=1$. *Assume that* $f(z)$ *and* $f(z+c)$ *share* a_3 *CM and*

$$\overline{E}_{1)}(a_j, f(z))\subseteq\overline{E}_{1)}(a_j, f(z+c)),\ j=1,2.$$

If $\delta(a, f)>0$ for some $a\neq a_4$, then $f(z)=f(z+c)$ for all $z\in\mathbb{C}$.

Obviously, Example B shows that condition $\delta(a, f)>0$ for some $a\neq a_4$ is necessary and sharp.

2. Some lemmas

Lemma 1 [9]: *Let* f *be a nonconstant meromorphic function on* $\mathbb{C}$. *If* $g=\dfrac{af+b}{cf+d}$, where $a,b,c,d\in S(f)$ and $ad-bc\not\equiv 0$, then $T(r,g)=T(r,f)+O(1)$.

Lemma 2 [10]: *Let* f *be a nonconstant entire function on* $\mathbb{C}$ *and* $f=e^h$. *Then* $\gamma(f)=\rho(h)$.

Lemma 3 [11, Corollary 1] *Let* f *be a nonconstant meromorphic function on* $\mathbb{C}$. *Let* $a_1,a_2,\ldots,a_q$ $(q\geq 3)$ *be* q *distinct small meromorphic functions of* f *on* $\mathbb{C}$. *Then the following holds*

$$(q-2)T(r,f)\leq\sum_{i=1}^{q}\overline{N}\left(r,\frac{1}{f-a_i}\right)+S(r,f).$$

Here, a meromorphic function a is small with respect to a meromorphic function f, we mean that $T(r,a)=o(T(r,f))$ as $r\to\infty$.

Lemma 4 [12] *Let* f *be a nonconstant meromorphic function and* $c\in\mathbb{C}$. *If* f *is of finite order, then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)=O\left(\frac{\log r}{r}T(r,f)\right)$$

for all r *outside of a subset* E *zero logarithmic density. If the hyper-order* $\gamma(f)$ *of* f *is less than one, then for each* $\varepsilon>0$, *we have*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)=o\left(\frac{T(r,f)}{r^{1-\gamma(f)-\varepsilon}}\right)$$

for all r *outside of a subset finite logarithmic measure.*

Lemma 5 [12, Theorem 2.1] *Let* $c\in\mathbb{C}$, *and let* f *be a meromorphic function of hyper-order* $\gamma(f)<1$ *such that* $\Delta_c f:=f_c-f\not\equiv 0$. *Let* $q\geq 2$ *and* $a_1(z),\ldots,a_q(z)$ *be distinct meromorphic periodic small functions of* f *with period* c. *Then,*

$$m(r,f)+\sum_{k=1}^{q}m(r,\frac{1}{f-a_k})\leq 2T(r,f)-N_{pair}(r,f)+S_1(r,f),$$

where

$$N_{pair}(r,f)=2N(r,f)-N(r,\Delta_c f)+N(r,\frac{1}{\Delta_c f})$$

3. Proof of Theorem

First of all, we put $F(z)=\dfrac{f(z)-a_3}{f(z)-a_4}\cdot\dfrac{a_1-a_4}{a_1-a_3}$ and put $b_1=1, b_2=c, b_3=0$ and $b_4=\infty$ where $c=\dfrac{a_2-a_3}{a_2-a_4}\cdot\dfrac{a_1-a_4}{a_1-a_3}$. Obviously, we have $c\neq 0,1,\infty$. By the assumption of the theorem, given meromorphic function and its shift share 0 CM and

$$\overline{E}_{1)}(1,F(z))\subseteq\overline{E}_{1)}(1,F(z+c));\ \overline{E}_{1)}(c,F(z))\subseteq\overline{E}_{1)}(c,F(z+c)). \quad (7)$$

In addition, by Lemma 1, we have

$\Theta(\infty,F)=1.$

We denote by $P(z)$ the canonical product of the poles of f.

Then, by Lemma 4, we have:

$$m\left(r,\frac{P(z+c)}{P(z)}\right)=S_1(r,F).$$

By $\Theta(\infty,F)=1$, and since above equation, we have

$$T\left(r,\frac{P(z+c)}{P(z)}\right)=S_1(r,F).$$

Since $F(z)$ and $F(z+c)$ share 0 CM, we get

$$\frac{F(z+c)}{F(z)}=e^{h(z)}\frac{P(z+c)}{P(z)}, \quad (8)$$

where h is an entire function. By Lemmas 1, 2, we have $\rho(h)=\gamma(f)=0.$

It follows from Lemma 4 and the first main theorem that:

$$T\left(r,e^{h(z)}\frac{P(z+c)}{P(z)}\right)=N\left(r,e^{h(z)}\frac{P(z+c)}{P(z)}\right)+m\left(r,e^{h(z)}\frac{P(z+c)}{P(z)}\right)+O(1)=S_1(r,F).$$

Put $\tau(z)=e^{h(z)}\dfrac{P(z+c)}{P(z)}$,

then τ is a small function with respect to F.

We now assume that $F(z)\neq F(z+c)$. It means that $\tau\not\equiv 1$ and we can rewrite (8) as follows

$$F(z+c)=\tau(z)F(z), \quad (9)$$

for all $z\in\mathbb{C}$. If $z_0\in\overline{E}_{1)}(b_i,F)$ $(i=1,2)$ then by (7) and (9), we get $\tau(z_0)=1$. Therefore,

$$\overline{N}_{1)}\left(r,\frac{1}{F(z)-b_i}\right)\leq\overline{N}\left(r,\frac{1}{\tau-1}\right)+O(1)=S_1(r,F).$$

It follows that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F(z)-b_i}\right)&=\overline{N}_{1)}\left(r,\frac{1}{F(z)-b_i}\right)+\overline{N}_{(2}\left(r,\frac{1}{F(z)-b_i}\right) \quad (10)\\&\leq\frac{1}{2}N\left(r,\frac{1}{F(z)-b_i}\right)+S_1(r,F)\\&\leq\frac{1}{2}T(r,F(z))+S_1(r,F),\quad \mathrm{j}=1,2.\end{aligned}$$

By definition of the deficiency and since (10), we get $\Theta(b_j,F)\geq\dfrac{1}{2}$, $j=1,2$

and hence $\Theta(b_1,F)+\Theta(b_2,F)+\Theta(\infty,F)\geq 2.$

It follows from the Second main theorem (Lemma 3) that $\Theta(b,F)=0$ for all $b\neq b_1,b_2,b_4$, i.e., $\delta(b,F)=0$ for all $b\neq b_1,b_2,b_4$.

For each $b\neq 0,\infty$, applying Lemma 5, we get

$$\begin{aligned}m(r,F(z))&+m\left(r,\frac{1}{F(z)}\right)+m\left(r,\frac{1}{F(z)-b}\right)\\&\leq 2T(r,F)-2N(r,F(z))+N(r,\Delta_cF)-N\left(r,\frac{1}{\Delta_cF}\right)+S_1(r,F)\\&=2T(r,F)-2N(r,F(z))+N(r,F(z))-N\left(r,\frac{1}{F(z)}\right)+S_1(r,F).\end{aligned}$$

This together with First main theorem implies that $T(r,F(z)) = N\left(r,\frac{1}{F(z)-b}\right) + S(r).$

It means $\delta(b,F) = 0$ for all $b \in \mathbb{C} \setminus \{b_3, b_4\}$.

From the above cases, we have $\delta(b,F) = 0, \forall b \neq b_4$. Using Lemma 1, we get $\delta(a,f) = 0$ for all values $a \in \mathbb{C} \cup \{\infty\} \setminus \{a_4\}$, which contradicts to the assumption. Therefore, we obtain $f(z) = f(z+c)$ for all $z \in \mathbb{C}$. Theorem 1 is proved.

4. Conclusion

Under an appropriate deficiency assumption, we showed that if a meromophic function f with hyper-order less than 1 partially sharing four small periodic functions with period c in the complex plane with its shift then f much be a periodic function with period c, i.e., $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

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