UNIQUENESS OF L - FUNCTIONS IN THE EXTENDED SELBERG CLASS

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ABSTRACT

Ritt's Second Theorem described polynomial solutions of the functional equation $P(f) = Q(g)$, where P, Q are polynomials. In this paper, using techniques of value distribution theory into account the special properties of L - functions, we describe solutions of the above equation for L - functions and a class of polynomials of Fermat-Waring type. Namely, use Lemma 2.1, Lemma 2.2, and Lemma 2.5, we study conditions to equations in the Theorem 1.1 have solutions on sets of L - functions in the extended Selberg class. Then we apply the obtained results from the Theorem 1.1, and use Lemma 2.3, Lemma 2.4, and Lemma 2.6 to study the uniqueness problem for L - functions sharing finite set in the Theorem 1.2.

Keyword: *Function equations; polynomials of Fermat-Waring type; shared sets; sets of zeros; L - functions.*

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TÍNH DUY NHẤT CỦA L – HÀM TRONG LỚP SELBERG MỞ RỘNG

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TÓM TẮT

Định lí Ritt thứ hai cho ta nghiệm đa thức của phương trình hàm $P(f) = Q(g)$, trong đó P, Q là đa thức. Trong bài báo này, sử dụng các kỹ thuật của lý thuyết phân phối giá trị có tính đến các thuộc tính đặc biệt của L - hàm, chúng tôi nghiên cứu phương trình hàm đa thức trên cho L - hàm và một lớp đa thức loại Fermat-Waring. Cụ thể, sử dụng Bổ đề 2.1, Bổ đề 2.2 và Bổ đề 2.5, chúng tôi nghiên cứu điều kiện để các phương trình trong Định lý 1.1 có nghiệm trên tập của L - hàm trong lớp Selberg mở rộng. Sau đó, chúng tôi áp dụng các kết quả thu được từ Định lý 1.1 và sử dụng Bổ đề 2.3, Bổ đề 2.4 và Bổ đề 2.6 để nghiên cứu vấn đề duy nhất cho các L - hàm nhận chung các tập hữu hạn trong Định lý 1.2.

Từ khóa: *Phương trình hàm; đa thức loại Fermat – Waring; tập chia sẻ; tập các không điểm; L - hàm*

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1 Introduction

In 1922, Ritt ([1]) studied functional equation $P(f) = Q(g)$, where P, Q are polynomials, and described it's polynomial solutions. Since the paper of Ritt ([1]), the functional equation $P(f) = Q(g)$, where P, Q are polynomials, has been investigated by many authors (see [2]- [6]). Pakovich [5] studied the functional equation $P(f) = Q(g)$, where P, Q are polynomials, and f, g are entire functions. Khoai-An-Hoa [6] considered the functional equation of the form $P(f) = Q(g)$, where P and Q are Yi's polynomials and then apply the obtained results to study the uniqueness problem for meromorphic functions sharing two subsets.

Now let us recall some basic notions. Let $\mathbb{C}$ denote the complex plane. By a *meromorphic function* we mean a meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the notations of Nevanlinna theory and of L-functions in the Selberg class(see [7-16]). In this paper, we discuss the Ritt's Second Theorem for L - functions and meromorphic functions. As a application, we present a uniqueness theorem for L - functions when the functions share values in a finite set.

Now let us describe the main results of the paper.

The Ritt's Second Theorem for L - functions is as following.

Theorem 1.1. *Let $a, b, c, c_1 \in \mathbb{C}$, $a \neq 0$, $b \neq 0$, $c \neq 0$, q be a positive integer.*

Then

1. The functional equation

$$x^q + a = c(y^q + b)(q \geq 3)$$

has a pair (L_1, L_2) of non-constant L - function solutions if and only if $L_1 = L_2$, $a = b$, $c = 1$.

2. The functional equation

$$\frac{1}{x^q + a} = \frac{c}{y^q + b} + c_1(q \geq 3)$$

has a pair (L_1, L_2) of non-constant L - function solutions if and only if $L_1 = L_2$, $a = b$, $c = 1$, $c_1 = 0$.

3. The functional equation

$$x^q y^q = a$$

has no non-constant L - function solutions (L_1, L_2).

Now let $a, b, c \in \mathbb{C}$, $a \neq 0$, $b \neq 0$, $c \neq 0$, q be a positive integer, and consider polynomials without multiple zero given by

$$P(z) = z^q + a,\ Q(z) = z^q + b, \qquad (1.1)$$

we obtain the following result.

Theorem 1.2. *Let L_1 and L_2 be two non-constant L - functions. Let P, Q be polynomials of the form (1.1), and S, T are respective sets of zeros of $P(z)$, $Q(z)$. Then $L_1 = L_2$ and $P = Q$ if one of the following conditions is satisfied:*

1. $q \geq 8$ and $\overline{E}_{L_1}(S) = \overline{E}_{L_2}(T)$;

2. $q \geq 3$ and $E_{L_1}(S) = E_{L_2}(T)$, $a \neq -1$, $b \neq -1$;

3. $q \geq 1$ and $E_{L_1}(S) = E_{L_2}(T)$, $a = b \neq -1$.

2 Some lemmas

We need some lemmas.

Lemma 2.1. **[10]** *Let f be a non-constant meromorphic function on $\mathbb{C}$ and let $a_1, a_2, ...,$ a_q be distinct points of $\mathbb{C} \cup \{\infty\}$. Then*

$$(q-2)T(r, f) \leq \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f - a_i}) + S(r, f),$$

where $S(r,f) = o(T(r,f))$ *for all* r, *except for a set of finite Lebesgue measure.*

Lemma 2.2. **[6]** *For any nonconstant meromorphic function* f,

$$N(r, \frac{1}{f'}) \leq N(r, \frac{1}{f}) + \overline{N}(r,f) + S(r,f).$$

Lemma 2.3. **[6]** *Let* f *and* g *be two non-constant meromorphic functions. If* $\overline{E}_f(1) = \overline{E}_g(1)$, *then one of the following three relations holds:*

1.

$$T(r,f) \leq N_2(r,f) + N_2(r,\frac{1}{f}) + N_2(r,g) + N_2(r,\frac{1}{g}) + 2(\overline{N}(r,f) + \overline{N}(r,\frac{1}{f})) + \overline{N}(r,g) + \overline{N}(r,\frac{1}{g}) + S(r,f) + S(r,g),$$

and the same inequality holds for $T(r,g)$;

2. $fg \equiv 1$;

3. $f \equiv g$.

Lemma 2.4. **[13]** *Let* L *be an non-constant* L *- function. Then*

1. $T(r,L) = \frac{d_L}{\pi} r \log r + O(r)$, *where* $d_L = 2\sum_{i=1}^{K} \lambda_i$ *be the degree of* L *- function and* K, λ_i *are respectively the positive integer and positive real number in the functional equation of the definition of* L *- functions;*

2. $N(r,\frac{1}{L}) = \frac{d_L}{\pi} r \log r + O(r)$, $N(r,L) = S(r,L)$.

Lemma 2.5. **[16]** *Let* $L_1, ..., L_N$ *be distinct non-constant* L *- functions. Then* $L_1, ..., L_N$ *are linearly independent over* $\mathbb{C}$.

Lemma 2.6. **[13]** *Let* L *be an non-constant* L *- functions and* $a \in \mathbb{C}$. *Then equation* $L = a$ *has infinitely many solutions.*

3 Proof of Theorems

Now we use the above Lemmas to prove the main result of the paper.

Proof of Theorem 1.1. 1. The sufficient condition of the theorem is easily seen. Now we show the necessary condition. Assume that

$$x^q + a = c(y^q + b)(q \geq 3) \tag{3.1}$$

has a pair (L_1, L_2) of non-constant L - function solutions. Then

$$L_1^q + a - cb = cL_2^q.$$

Suppose $a - cb \neq 0$. Then, by Lemma 2.1, and note that $\overline{N}(r,L_1) = S(r,L_1)$, $\overline{N}(r,L_2) = S(r,L_2)$, we obtain

$$qT(r,L_1) + S(r,L_1) = T(r,L_1^q)$$

$$\leq \overline{N}(r,L_1) + \overline{N}(r,\frac{1}{L_1}) + \overline{N}(r,\frac{1}{L_1^q + a - bc}) + S(r,L_1)$$

$$\leq \overline{N}(r,\frac{1}{L_1}) + \overline{N}(r,\frac{1}{L_2}) + S(r,L_1) \leq T(r,L_1) + T(r,L_2) + S(r,L_1).$$

Similarly

$$qT(r,L_2) + S(r,L_2) \leq T(r,L_2) + T(r,L_1) + S(r,L_2).$$

Therefore

$$q(T(r,L_1) + q(T(r,L_2)) \leq 2(T(r,L_1) + T(r,L_2)) + S(r,L_1) + S(r,L_2),$$

$$(q-2)(T(r,L_1) + T(r,L_2) \leq S(r,L_1) + S(r,L_2).$$

This is a contradiction to the assumption that $q \geq 3$. So $a - cb = 0$. Then $L_1^q = cL_2^q$. From this $L_2 = tL_1, t^q c = 1$. Applying Lemma 2.5 we have $L_1 = L_2$ and therefore $t = 1$, $a = b$, $c = 1$.

2. The sufficient condition of the theorem is easily seen. Now we show the necessary condition. Assume that

$$\frac{1}{x^q+a}=\frac{c}{y^q+b}+c_1 \qquad (3.2)$$

has a pair (L_1, L_2) of non-constant L - function solutions. Then

$$\frac{1}{L_1^q+a}=\frac{c}{L_2^q+b}+c_1.$$

We shall prove that $c_1=0$. Suppose, to the contrary, $c_1\neq 0$. Note that when considering L-functions, these functions have only one possible pole at $s=1$. Write

$$L_1(s)=\frac{L_{10}(s)}{(s-1)^{m_1}},\ L_2(s)=\frac{L_{20}(s)}{(s-1)^{m_2}},$$

$$m_1\geq 0, m_2\geq 0,$$

where $L_i(s)$, and $(s-1)^{m_i}, i=1,2$, has no common zero. From this and (3.2) we get

$$\frac{(s-1)^{qm_1}}{L_{10}^q(s)+a(s-1)^{qm_1}}=\frac{c(s-1)^{qm_2}}{L_{20}^q(s)+b(s-1)^{qm_2}}+c_1. \qquad (3.3)$$

By $c_1\neq 0$ we have

$$m_1.m_2=0. \qquad (3.4)$$

We recall that $P(x)=x^q+a$, $Q(y)=y^q+b$. Put $R(y)=Q(y)+\frac{c}{c_1}$. Suppose that $R(z)$ has distinct zeros $e_1, e_2, ..., e_k$ with respective multiplicities $l_1, l_2, ..., l_k$, $1\leq k\leq q$, so that $l_1+\cdots+l_k=q$. Then we get

$$\frac{Q(L_2)}{c_1P(L_1)}=Q(L_2)+\frac{c}{c_1}=R(L_2)$$

$$=(L_2-e_1)^{l_1}...(L_2-e_k)^{l_k},$$

$$\frac{(s-1)^{q(m_1-m_2)}(L_{20}^q(s)+b(s-1)^{qm_2})}{c_1(L_{10}^q(s)+a(s-1)^{qm_1})}=(L_2-e_1)^{l_1}...(L_2-e_k)^{l_k}. \qquad (3.5)$$

Note that L_2-e_i, $i=1,...,k$, always has zeros. Then, if s_0 be a zero of L_2-e_i, then

$$\frac{Q(L_2(s_0))}{c_1P(L_1(s_0))}=0,\ Q(L_2(s_0))+\frac{c}{c_1}=0,$$

$$\frac{(s_0-1)^{q(m_1-m_2)}(L_{20}^q(s_0)+b(s_0-1)^{qm_2})}{c_1(L_{10}^q(s_0)+a(s_0-1)^{qm_1})}=0. \qquad (3.6)$$

Since (3.6) and $c, c_1\neq 0$ we get $Q(L_2(s_0))=-\frac{c}{c_1}\neq 0$. Therefore $L_{20}^q(s_0)+b(s_0-1)^{qm_2}\neq 0$. So $(s_0-1)^{q(m_1-m_2)}=0$. It follows that $m_1>m_2$. Consider (3.5). Write

$$P(z)=(z-a_1)...(z-a_q),$$

$$P(L_1)=(L_1-a_1)...(L_1-a_q).$$

From (3.5), (3.6) and note that L_1-a_i, $i=1,...,q$, always has zeros we have L_2 has pole at $s=1$. Thus $m_2>0$. By $m_1>m_2>0$ and (3.4), (3.5) we have a contradiction.

Thus $c_1=0$. Therefore $P(L_1)=CQ(L_2)$. Applying Part i) we get $L_1=L_2$, $a=b$, $c=1$, $c_1=0$.

3. Suppose, to the contrary, functional equation

$$x^qy^q=a$$

has a non-constant L - function solution (L_1, L_2). Then

$$L_1^qL_2^q=a.$$

Note that L_1, L_2 have only one possible pole at $s=1$. Onthe other hand L_1, L_2 have infinitely zeros. Therefore, there is a $s_0\neq 1$ such that $L_1(s_0)=0$. So $a=0$. A contradiction to assumption that $a=0$.

Proof of Theorem 1.2.

Proof of Part 1. Set $F=\frac{L_1^q}{-a}$, $G=\frac{L_2^q}{-b}$, $T(r)=T(r,f)+T(r,g)$, $S(r)=S(r,f)+S(r,g)$. By $\overline{E}_{L_1}(S)=\overline{E}_{L_2}(S)$ it follow that $\overline{E}_F(1)=\overline{E}_G(1)$. Then, applying Lemma 2.3 to the F, G we consider the following cases:

Case 1.

$$T(r,F) \leq N_2(r,F) + N_2(r,\frac{1}{F})$$
$$+N_2(r,G)+N_2(r,\frac{1}{G})+2(\overline{N}(r,F)+\overline{N}(r,\frac{1}{F}))$$
$$+\overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+S(r),$$
$$T(r,G) \leq N_2(r,F) + N_2(r,\frac{1}{F})$$
$$+N_2(r,G)+N_2(r,\frac{1}{G})+2(\overline{N}(r,G)+\overline{N}(r,\frac{1}{G}))$$
$$+\overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+S(r). \quad (3.7)$$

Noting that

$$N(r,L_1) = S(r,L_1),\ N(r,L_2) = S(r,L_2),$$
$$\overline{N}(r,F) = \overline{N}(r,L_1) = S(r,L_1),$$
$$\overline{N}(r,F) = \overline{N}(r,L_1) = S(r,L_1),$$
$$N_2(r,F) = 2N(r,L_1) = S(r,L_1),\ N_2(r,G)$$
$$= 2N(r,L_2) = S(r,L_2),\ N_2(r,\frac{1}{F}) = 2\overline{N}(r,\frac{1}{L_1})$$
$$N_2(r,\frac{1}{G}) = 2\overline{N}(r,\frac{1}{L_2}). \quad (3.8)$$

From (3.7), (3.8) we obtain

$$T(r,F) = qT(r,L_1)$$
$$\leq 4\overline{N}(r,\frac{1}{L_1}) + 3\overline{N}(r,\frac{1}{L_2}) + S(r)$$
$$\leq 4T(r,L_1) + 3T(r,L_1) + S(r),$$
$$T(r,G) = qT(r,L_2)$$
$$\leq 3\overline{N}(r,\frac{1}{L_1}) + 4\overline{N}(r,\frac{1}{L_2}) + S(r)$$
$$\leq 3T(r,L_1) + 4T(r,L_1) + S(r).$$

Therefore

$$qT(r) \leq 7T(r) + S(r),\ (q-7)T(r) \leq S(r).$$

This is a contradiction to the assumption that $q \geq 8$.

Case 2. $F.G = 1$. This mean $\frac{L_1^q}{-a}\frac{L_2^q}{-b} = 1$ or $L_1^q L_2^q = A, A \neq 0$. Applying Theorem 1.1 we obtain a contradiction.

Case 3. $F = G$. This mean $\frac{L_1^q}{-a} = \frac{L_2^q}{-b}$ or $L_1^q = CL_2^q$. Applying Theorem 1.1 we get $L_1 = L_2$ and therefore $a = b$.

Proof of Part 2. We first prove that $L_1 + a = c(L_2 + b)$. We consider the following cases:

Case 4. $L_1(s), L_2(s)$ are both entire functions and share the respective sets S, T CM, where $S = \{a_1, ..., a_q\}$ with $P(z) = (z - a_1)...(z - a_q)$, and $T = \{b_1, ..., b_q\}$ with $Q(z) = (z - b_1)...(z - b_q)$. Then, we obtain an entire function

$$l(s) = \frac{(L_1 - a_1)...(L_1 - a_q)}{(L_2 - b_1)...(L_2 - b_q)},$$

with $l(s) \neq 0, \infty$ for all $s \in \mathbb{C}$. By the First Fundamental Theorem,

$$T(r,\frac{1}{L_2 - b_i}) = T(r,L_2) + O(1), i = 1,...,q.$$

Denote the order of a meromorphic function f by $\rho(f)$, then it follows that

$$\rho(\frac{1}{L_2 - b_i}) = \rho(L_2) = 1.$$

Moreover,

$$\rho(L_1 - a_i) = \rho(L_1) = 1, i = 1,...,q.$$

Since the order of a finite product of functions of finite order is less then or equal to the maximum of the order of these factors (see [17]), we have $\rho(l) \leq 1$. This implies that $l(s)$ is of the form $l(s) = e^{As+B}$ where A, B are constants. Since

$$\lim_{s\to+\infty} L_i(s) = 1,$$

we get

$$\lim_{s\to+\infty} l(s) = \lim_{s\to+\infty} \frac{(L_1 - a_1)...(L_1 - a_q)}{(L_2 - b_1)...(L_2 - b_q)} = \frac{P(1)}{Q(1)}.$$

This implies that $A = 0$, that is, $l(s) = c$.

Case 5. $L_1(s)$ or $L_2(s)$ has a pole at $s = 1$ with multiplicity $m_1(\geq 0)$ or $m_2(\geq 0)$, respectively. Set

$$l(s) = \frac{(s-1)^m (L_1 - a_1)...(L_1 - a_q)}{(L_2 - b_1)...(L_2 - b_q)},$$

where $m = q(m_2 - m_1)$ is an integer. We use the arguments similar to the *Case 4.* So we conclude that

$$l(s) = e^{As+B}, \text{ where A, B are constants.}$$

Moreover

$$\lim_{s\to+\infty}(s-1)^{-m}e^{As+B} = \lim_{s\to+\infty}(s-1)^{-m}l(s)$$

$$= \lim_{s\to+\infty}\frac{(L_1 - a_1)...(L_1 - a_q)}{(L_2 - b_1)...(L_2 - b_q)} = \frac{P(1)}{Q(1)}.$$

So $A = 0, m = 0$, that is, $l(s) = c$. Thus $L_1 + a = c(L_2 + b)$. Applying Theorem 1.1 we get $L_1 = L_2, a = b$.

Proof of Part 3. We use the arguments similar to the *Proof of Part 2* and then applying Theorem 1.1 we get $L_1 = L_2$.

4 Conclusion

We study conditions to equations:

$$x^q + a = c(y^q + b), \ \frac{1}{x^q + a} = \frac{c}{y^q + b} + c_1,$$

$$x^q y^q = a$$

have solutions on sets of L - functions in the extended Selberg class. Since we give some sufficient conditions for a finite set S to be a uniqueness range set of L - functions in the extended Selberg class. Namely, let $P(z) = z^q + a$, $Q(z) = z^q + b$ and let S, T are respective sets of zeros of $P(z)$, $Q(z)$. Then $L_1 = L_2$ and $P = Q$ if one of the following conditions is satisfied:

1. $q \geq 8$ and $\overline{E}_{L_1}(S) = \overline{E}_{L_2}(T)$;

2. $q \geq 3$ and $E_{L_1}(S) = E_{L_2}(T)$, $a \neq -1$, $b \neq -1$;

3. $q \geq 1$ and $E_{L_1}(S) = E_{L_2}(T)$, $a = b \neq -1$.

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