ISSN 2354-0575

THE INDUCTIVE METHOD FOR INFINITE-TIME RUIN PROBABLILITY IN QUOTA- (*α, β*) REINSURANCE MODEL

Nguyen Quang Chung
Hung Yen University of Techonology and Education

Received: 10/01/2020
Revised: 15/02/2020
Accepted for publication: 20/02/2020

Abstract:

In this article, we investigate a risk model with a quota-(α, β) reinsurance contract. The premium process and claim process are assumed to be independent sequences of indentically distributed random variables. Using inductive method, we obtain upper bound of infinite-time ruin probability of an insurance company.

Keywords: *inductive method, infinite-time ruin probability, reinsurance contract, recursive equation, upper bound.*

1. Introduction

The problem of ruin has a long history in risk theory, one of the simplest risk model was shown by Dickson [1] where the insurer's premium income per unit time is 1 and the claim process is a sequence of independent and identically distributed random variables. The authors [2]-[9] introduced the risk models with interest. Particularly, Dam and Chung [10], [11] presented a risk model with a reinsurance contract effect included.

In this article, we consider a risk model which has a surplus process of the insurance company:

$$U_n = u_0 + \alpha \sum_{i=1}^{n} X_i + \beta \sum_{i=1}^{n} Y_i ,$$

n = 1, 2, ..., (1.1)

where

- u_0 is the initial capital,
- X_n and Y_n are the premium income and claim size in the th period, respectively,
- α and β $(\alpha, \beta \in [0,1])$ are division rations to share the premiums and claims between the insurer and the reinsurer.

We denote the finite-time ruin probability and the infinite-time ruin probability of the surplus process (1.1) by

$$\psi_n(u_0, \alpha, \beta) = \mathbb{P}\Big(\bigcup_{i=1}^{n}(U_i \le 0)\Big),$$

n =1, 2, ...

and

$$\psi(u_0, \alpha, \beta) = \mathbb{P}\Big(\bigcup_{i=1}^{\infty}(U_i \le 0)\Big).$$

Dam and Chung [10] estimated for ruin probabilities in the risk model (1.1) by Martingale method, this upper bound has an exponential form. Our article will use the inductive method to estimate for ruin probability in the risk model (1.1). It is very popular in risk theory research, see ([3],[4],[8],[9],[11],[12]).

The rest of the article is organized as follows. In Section 1, we will introduce the article's risk model. The upper bound for the infinite-time ruin probability will be given in Section 2.

2. Main Results and Discussions

Firstly, we present the following lemma:

***Lemma 2.1**. If* $\alpha\mathbb{E}(X_1) > \beta\mathbb{E}(Y_1)$ *and* $\mathbb{P}(\beta Y_1 - \alpha X_1) > 0$ *for any* (α, β) *then there exists a unique positive root,* $R(\alpha, \beta)$, *such that*

$$\mathbb{E}\left(e^{R(\alpha,\beta)(\beta Y_1 - \alpha X_1)}\right) = 1. \tag{2.1}$$

Proof. See Dam and Chung [10].

We now denote distribution functions of X_1 and Y_1 by $H(x)$ and $F(y)$, respectively. The following theorem gives a recursive equation for $\psi_n(u_0, \alpha, \beta)$. The theorem's proof is similar to the one in [4],[9],[11],[12].

***Theorem 2.2**. For any given* u_0 *and a pair* $(\alpha, \beta)(\alpha, \beta \in (0,1))$, *we have:*

$$\psi_{n+1}(u_0, \alpha, \beta) = \int_0^\infty \int_0^{\frac{1}{\beta}(u_0 + \alpha x)} \psi_n(u_0 + \alpha x - \beta y, \alpha, \beta) dF(y) H(x) + \int_0^\infty \overline{F}\Big(\frac{1}{\beta}(u_0 + \alpha x)\Big) dH(x),$$

$n = 1, 2, ...$ (2.2)

In particular,

$$\psi_1(u_0, \alpha, \beta) = \int_0^\infty \overline{F}\Big(\frac{1}{\beta}(u_0 + \alpha x)\Big) dH(x) \tag{2.3}$$

where $\overline{F}(y) = 1 - F(y)$.

***Proof*.** For $n = 1,2, \ldots$

$$\psi_{n+1}(u_0,\alpha,\beta) = \mathbb{P}\left(\bigcup_{i=1}^{n}(U_i \le 0)\right)$$

$$= \int_0^{\infty}\int_0^{\infty} \mathbb{P}\left(\bigcup_{i=1}^{n+1}(U_i \le 0) | X_1 = x, Y_1 = y\right)dF(y)dH(x)$$

$$= \int_0^{\infty}\int_0^{\frac{1}{\beta}(u_0+\alpha x)} \mathbb{P}\left(\bigcup_{i=1}^{n+1}(U_i \le 0) | X_1 = x, Y_1 = y\right)dF(y)dH(x)$$

$$+\int_0^{\infty}\int_{\frac{1}{\beta}(u_0+\alpha x)}^{\infty} \mathbb{P}\left(\bigcup_{i=1}^{n+1}(U_i \le 0) | X_1 = x, Y_1 = y\right)dF(y)dH(x) \quad (2.4)$$

If $y \ge \frac{1}{\beta}(u_0+\alpha x)$ then the insurer's ruin occurs at period $n = 1$. Hence

$$\mathbb{P}(U_1 \le 0 | X_1 = x, Y_1 = y) = 1.$$

Which implies that

$$\mathbb{P}\left(\bigcup_{i=1}^{n+1}(U_i \le 0) | X_1 = x, Y_1 = y\right) = 1. \quad (2.5)$$

If $0 \le y < \frac{1}{\beta}(u_0+\alpha x)$ then the insurer's ruin does not occurs at period $n = 1$. i.e.

$$\mathbb{P}(U_1 \le 0 | X_1 = x, Y_1 = y) = 0$$

since

$$\mathbb{P}\left(\bigcup_{i=1}^{n+1}(U_i \le 0) | X_1 = x, Y_1 = y\right)$$

$$= \mathbb{P}\left(\bigcup_{i=2}^{n+1}(U_i \le 0) | X_1 = x, Y_1 = y\right)$$

$$= \psi_n(u_0+\alpha x - \beta y,\alpha,\beta). \quad (2.6)$$

Plugging (2.5) and (2.6) into (2.4), we have

$$\psi_{n+1}(u_0,\alpha,\beta) = \int_0^{\infty}\int_0^{\frac{1}{\beta}(u_0+\alpha x)} \psi_n(u_0+\alpha x-\beta y,\alpha,\beta)dF(y)H(x)$$

$$+\int_0^{\infty} \overline{F}\left(\frac{1}{\beta}(u_0+\alpha x)\right)dH(x).$$

Further, (2.3) follows from

$$\psi_1(u_0,\alpha,\beta) = \mathbb{P}(U_1 \le 0)$$

$$= \int_0^{\infty}\int_0^{\infty} \mathbb{P}(u_0+\alpha X_1 - \beta Y_1 \le 0 | X_1 = x, Y_1 = y)dF(y)dH(x)$$

$$= \int_0^{\infty}\int_0^{\frac{1}{\beta}(u_0+\alpha x)} \mathbb{P}(u_0+\alpha X_1 - \beta Y_1 \le 0 | X_1 = x, Y_1 = y)dF(y)dH(x)$$

$$+\int_0^{\infty}\int_{\frac{1}{\beta}(u_0+\alpha x)}^{\infty} \mathbb{P}(u_0+\alpha X_1 - \beta Y_1 \le 0 | X_1 = x, Y_1 = y)dF(y)dH(x)$$

$$= \int_0^{\infty} \overline{F}\left(\frac{1}{\beta}(u_0+\alpha x)\right)dH(x).$$

This ends the proof of Theorem 2.1.

Equation (2.2) is called recursive equation for the finite-time ruin probability of the insurer.

We next use the recursive equation for the finite-time ruin probability to derive an inequality for the ultimate (infinite-time) ruin probability.

Theorem. 2.3. *Assuming that the surplus process given in (1.1) satisfy the conditions of Lemma 2.1. For any* $(\alpha,\beta)(\alpha,\beta \in (0,1))$ *then*

$$\psi(u_0,\alpha,\beta) \le \gamma e^{-u_0 R(\alpha,\beta)} \quad (2.7)$$

where

$$\gamma^{-1} = \inf_{z\ge 0}\frac{\int_z^{\infty} e^{BR(\alpha,\beta)y}dF(y)}{e^{BR(\alpha,\beta)z}\overline{F}(z)}.$$

Proof.

$$\overline{F}(z) = \left\{\frac{\int_z^{\infty} e^{\beta R(\alpha,\beta)y}dF(y)}{e^{\beta R(\alpha,\beta)z}\overline{F}(z)}\right\}^{-1} e^{-\beta R(\alpha,\beta)z}\int_z^{\infty} e^{\beta R(\alpha,\beta)y}dF(y)$$

$$\le \gamma e^{-\beta R(\alpha,\beta)z}\int_z^{\infty} e^{\beta R(\alpha,\beta)y}dF(y) \quad (2.8)$$

$$\le \gamma e^{-\beta R(\alpha,\beta)z}\int_0^{\infty} e^{\beta R(\alpha,\beta)y}dF(y)$$

$$= \gamma e^{-\beta R(\alpha,\beta)z}\mathbb{E}(e^{\beta R(\alpha,\beta)Y_1}). \quad (2.9)$$

Replacing z by $\frac{1}{\beta}(u_0+\alpha x)$ in (2.9) and using (2.3), we have

$$\psi_1(u_0,\alpha,\beta)$$

$$\le \gamma\mathbb{E}(e^{\beta R(\alpha,\beta)Y_1})\int_0^{\infty} e^{-R(\alpha,\beta)(u_0+\alpha x)}dH(x)$$

$$= \gamma e^{-R(\alpha,\beta)u_0}\mathbb{E}(e^{R(\alpha,\beta)(\beta Y_1-\alpha X_1)}) = \gamma e^{-R(\alpha,\beta)u_0}. \quad (2.10)$$

Under an inductive hypothesis, we assume that

$$\psi_n(u_0,\alpha,\beta) \le \gamma e^{-R(\alpha,\beta)u_0}. \quad (2.11)$$

Clearly, for $u_0+\alpha x-\beta y > 0$ replacing u_0 by $u_0+\alpha x-\beta y$ in (2.11), we get

$$\psi_n(u_0+\alpha x-\beta y,\alpha,\beta) \le \gamma e^{-R(\alpha,\beta)(u_0+\alpha x-\beta y)}. \quad (2.12)$$

From (2.2), (2.12) and z replaced by $\frac{1}{\beta}(u_0+\alpha x)$ in (2.8), we have

$$\psi_{n+1}(u_0,\alpha,\beta)$$

$$\le \int_0^{\infty}\int_0^{\frac{1}{\beta}(u_0+\alpha x)} \gamma e^{-R(\alpha,\beta)(u_0+\alpha x-\beta y)}dF(y)dH(x)$$

$$+\int_0^{\infty}\int_{\frac{1}{\beta}(u_0+\alpha x)}^{\infty} \gamma e^{-R(\alpha,\beta)(u_0+\alpha x-\beta y)}dF(y)dH(x)$$

$$= \int_0^\infty \int_0^\infty \gamma e^{-R(\alpha,\beta)(u_0+\alpha x-\beta y)} dF(y)dH(x)$$
$$= \gamma e^{-R(\alpha,\beta)u_0} \mathbb{E}\left(e^{R(\alpha,\beta)(\beta Y_1-\alpha X_1)}\right)$$
$$= \gamma e^{-R(\alpha,\beta)u_0}. \quad (2.13)$$

Since, (2.7) follows by letting $n \to \infty$ in (2.13). This completes the proof.

Remark:

1). We note that $0 \leq \gamma \leq 1$ since for any $z \geq 0$

$$\gamma^{-1} = \inf_{z\geq 0} \frac{\int_z^\infty e^{BR(\alpha,\beta)y} dF(y)}{e^{BR(\alpha,\beta)z}\overline{F}(z)}$$
$$\geq \inf_{z\geq 0} \frac{\int_z^\infty e^{BR(\alpha,\beta)z} dF(y)}{e^{BR(\alpha,\beta)z}\overline{F}(z)} = 1$$

The upper bound in Theorem 2.3 is less than the upper bound in [10], namely

$$\gamma e^{-R(\alpha,\beta)u_0} \leq e^{-R(\alpha,\beta)u_0}.$$

2.) If $\alpha = \beta$ then the upper bound in Theorem 2.3 is the upper bound in [11].

3. Conclusion

Our article established the upper bound for the infinite-time ruin probability of an insuarance company. The results are basic to study the ruin probabilities in risk models with *m*- dependence assumptions. The topic of the day will be presented in our future articles.

References

[1]. D.C.M. Dickson. *Insurance Risk and Ruin*. Cambridge University Presss, 2006.

[2]. H. Yang. Non-exponential Bounds for Ruin Probability with Interest Effect Included. *Scandinavian Actuarial Journal*, **1**: 66-79, 1999.

[3]. J. Cai. Ruin Probabilities with Dependent Rates of Interest. *Journal of Applied Probability*, **39(2)**:312-323, 2002.

[4]. J. Cai, D.C.M. Dickson. Ruin probabilities with a Markov chain interest model. *Insurance: Mathematics and Economics*, **35**: 513-525, 2004.

[5]. Q. Tang. The Ruin Probability of a Discrete Time Risk Model under Constant Interest Rate with Heavy Tails. *Scandinavian Actuarial Journal*. 2004, **3**: 229-240.

[6]. X. Wei, Y. Hu. Ruin probabilities for discrete time risk models with stochastic rates of interest. *Statistics & Probability Letters*, **78**: 707-715, 2008.

[7]. C. Weng, Y. Zhang, K.S. Tan. Ruin probabilities in a discrete time risk model with dependent risks of heavy tail. *Scandinavian Actuarial Journal*, **3**: 205-218, 2009.

[8]. X. Lin, Z. Dongjin, Z. Yanru. Minimizing Upper Bound of Ruin Probability Under Discrete Risk Model with Markov Chain Interest Rate. *Communications in Statistics - Theory and Methods*, **44(4)**: 810-822, 2015.

[9]. P.D. Quang. Ruin Probability in a Generalised Risk Process under Rates of Interest with Homogenous Markov Chain. *East Asian Journal on Applied Mathematics*, **4(3)**:283-300, 2014.

[10]. B.K. Dam, N.Q. Chung. The Martingale Method for Probability of Ultimate Ruin Under Quota Reinsurance Model. *Journal of Statistics & Application Probability*, **5(3)**:411-419, 2016.

[11]. B.K. Dam, N.Q. Chung. On Finite- Time Ruin Probabilities in a Risk Model under Quota Share Reinsurance Contract. *Applied Mathematical Sciences*, **11(35)**:2609-2629, 2017.

[12]. J.Cai and D.C.M. Dickson. Upper bounds for ultimate ruin probabilities in the Sparre Andersen model with interest. *Insurance: Mathematics and Economics*, **32**: 61-71, 2003.

PHƯƠNG PHÁP QUY NẠP CHO XÁC SUẤT PHÁ SẢN VỚI THỜI GIAN VÔ HẠN TRONG MÔ HÌNH RỦI RO TÁI BẢO HIỂM QUOTA- (*α,β*)

Tóm tắt:

Bài báo nghiên cứu mô hình rủi ro tái bảo hiểm quota- (α,β), giả thiết dãy thu bảo hiểm và chi trả bảo hiểm là dãy các biến ngẫu nhiên độc lập cùng phân phối. Sử dụng phương pháp quy nạp, bài báo đã thiết lập được chặn trên cho xác suất phá sản với thời gian vô hạn của công ty bảo hiểm.

Từ khóa: *Phương pháp quy nạp, xác suất phá sản, hợp đồng tái bảo hiểm, phương trình đệ quy, chặn trên.*