# A REGULARIZATION METHOD FOR BACKWARD PARABOLIC EQUATIONS WITH TIME-DEPENDENT COEFFICIENTS

**Nguyen Van Duc** (1), **Tran Hoai Bao** (2)
[1] *School of Natural Sciences Education, Vinh University, Vinh City, Vietnam*
[2] *Ha Tinh High School for the Gifted, Ha Tinh City, Vietnam*
Received on 19/5/2019, accepted for publication on 15/7/2019

**Abstract:** Let $H$ be a Hilbert space with the norm $\|\cdot\|$ and $A(t), (0 \leqslant t \leqslant T)$ be positive self-adjoint unbounded operators from $D(A(t)) \subset H$ to $H$. In the paper, we propose a regularization method for the ill-posed backward parabolic equation with time-dependent coefficients

$$\begin{cases} u_t + A(t)u = 0, \quad 0 < t < T, \\ \|u(T) - f\| \leqslant \varepsilon, \quad f \in H, \varepsilon > 0. \end{cases}$$

A priori and a posteriori parameter choice rules are suggested which yield errors estimates of Hölder type. Our errors estimates improve the related results in [4].

## 1 Introduction

Let $H$ be a Hilbert space equipped the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, $A(t)$ $(0 \leqslant t \leqslant T) : D(A(t)) \subset H \to H$ be positive self-adjoint unbounded operators on $H$. Let $f$ in $H$ and $\varepsilon$ be a given positive number. We consider the backward parabolic problem of finding a function $u : [0, T] \to H$ such that

$$\begin{cases} u_t + A(t)u = 0, \quad 0 < t < T, \\ \|u(T) - f\| \leqslant \varepsilon. \end{cases} \tag{1}$$

This problem is well-known to be severely ill-posed [8], [9]. Therefore, the stability estimates and the regularization methods [11] are required.

It was proved in [4] that, if $u(t)$ is a solution of the equation $u_t + A(t)u = 0,\ 0 < t < T$, then there exists a non-negative function $\nu(t)$ on $[0, T]$ such that

$$\|u(t)\| \leqslant c\|u(T)\|^{\nu(t)}\|u(0)\|^{1-\nu(t)}, \quad \forall t \in [0, T], \tag{2}$$

where $c$ is a positive constant. Furthermore, a priori and a posteriori parameter choice rules were suggested yielding the errors estimates of Hölder type with an order $\frac{\nu(t)}{2}$. In this paper, we investigate the regularization of the problem (1). The main tools will be based on the method of non-local boundary value problems [1]-[6] and the parameter choice rules of a priori and a posteriori. We then proved that these parameter choice rules yield the errors estimates of Hölder type with an order $\nu(t)$. This is an improvement of the related results in [4].

[1] Email:ducnv@vinhuni.edu.vn (N. V. Duc)

## 2 Preliminaries

Let us recall the following result from Theorem 2.5 in [4].
Suppose that

(i) $A(t)$ is a self-adjoint operator for each $t$, and $u(t)$ belongs to domain of $A(t)$

(ii) If $u(t)$ is a solution of the equation

$$Lu := \frac{du}{dt} + A(t)u = 0, \quad 0 < t \leqslant T$$

then for some non-negative constants $k, c$, it holds that

$$-\frac{d}{dt}\langle A(t)u(t), u(t)\rangle \geqslant 2\|A(t)u\|^2 - c\langle (A(t)+k)u(t), u(t)\rangle .$$

Let $a_1(t)$ be a continuous function on $[0,T]$ satisfying $a_1(t) \leqslant c, \forall t \in [0,T]$ and

$$-\frac{d}{dt}\langle A(t)u(t), u(t)\rangle \geqslant 2\|A(t)u\|^2 - a_1(t)\langle (A(t)+k)u(t), u(t)\rangle .$$

For all $t \in [0,T]$, let

$$a_2(t) = \exp\left(\int_0^t a_1(\tau)d\tau\right), \ a_3(t) = \int_0^t a_2(\xi)d\xi,$$
$$\nu(t) = \frac{a_3(t)}{a_3(T)}. \tag{3}$$

Then

$$\|u(t)\| \leqslant e^{kt-kT\nu(t)}\|u(T)\|^{\nu(t)}\|u(0)\|^{1-\nu(t)}, \quad \forall t \in [0,T]. \tag{4}$$

## 3 Main results

In this section, we make the following assumptions for the operators $A(t)$ [12; pp. 134-135].

$(H_1)$ For $0 \leqslant t \leqslant T$, the spectrum of $A(t)$ is contained in a sectorial open domain

$$\sigma(A(t)) \subset \Sigma_\omega = \{\lambda \in \mathbb{C};\ |\arg\lambda| < \omega\},\ 0 \leqslant t \leqslant T, \tag{5}$$

with some fixed angle $0 < \omega < \frac{\pi}{2}$, and the resolvent satisfies the estimate

$$\|(\lambda - A(t))^{-1}\| \leqslant \frac{M}{|\lambda|}, \quad \lambda \notin \Sigma_\omega,\ 0 \leqslant t \leqslant T, \tag{6}$$

with some constant $M \geqslant 1$.

$(H_2)$ The domain $D(A(t))$ is independent of $t$ and $A(t)$ is strongly continuously differentiable [10; p. 15].

$(H_3)$ For all $t \in [0,T]$, $A(t)$ is a positive self-adjoint unbounded operator and if $u(t)$ is a solution of the equation $Lu = \dfrac{du}{dt} + A(t)u = 0, \quad 0 < t \leqslant T$, then there are a non-negative constant $k$ and a continuous function on $[0,T]$, $a_1(t)$ such that

$$-\frac{d}{dt}\langle A(t)u(t), u(t)\rangle \geqslant 2\|A(t)u\|^2 - a_1(t)\langle (A(t)+k)u(t), u(t)\rangle . \tag{7}$$

**Remark 3.1.** (See [4]) If assumptions $(H_1)$ and $(H_2)$ are satisfied, then

$$\|A(t)(A(t)^{-1} - A(s)^{-1})\| \leqslant N|t-s|,\ 0 \leqslant s,t \leqslant T, \tag{8}$$

for some constant $N > 0$.

To regularize (1), following Fritz John [7], we should impose some prescribed bound for $u(0)$. Namely, in this section we suppose that there is a positive constant $E$ such that

$$\|u(0)\| \leqslant E. \tag{9}$$

Now, let

$$B(t) = \begin{cases} A(-t), \text{ if } -T \leqslant t \leqslant 0, \\ A(t), \text{ if } 0 < t \leqslant T. \end{cases} \tag{10}$$

Then $B(t) = B(-t), \forall t \in [-T, T]$. Furthermore, $B(t),\ (-T \leqslant t \leqslant T)$ are also positive self-adjoint unbounded operators, the domain $D(B(t))$ is independent of $t$ and $B(t),\ (-T \leqslant t \leqslant T)$ also satisfy the conditions (5), (6) and (8).

In this paper, the ill-posed parabolic equation backward in time (1) subjects to the constraint (9), is regularized by the problem

$$\begin{cases} v_t + B(t)v = 0, \quad -T < t < T, \\ \alpha v(-T) + v(T) = f, \end{cases} \tag{11}$$

where $\alpha$ is a positive number.

From now on, for clarity, we denote the solution of (1), (9) by $u(t)$, the solution of the problem (11) by $v(t)$ and $z(t) = u(t) - v(t),\ \forall t \in [0, T]$. We have $z(t)$ is the solution of the problem

$$\begin{cases} z_t + A(t)z = 0, \quad 0 < t < T, \\ z(0) = u(0) - v(0). \end{cases} \tag{12}$$

**Theorem 3.2.** *The problem (11) is well-posed.*

*Proof.* The proof of this theorem is an application of Lemma 3.3 and Lemma 3.4 below. □

**Lemma 3.3.** *If $v(t)$ is a solution of (11), then*

$$\alpha^2\|v(-T)\|^2 + (2\alpha+1)\|v(T)\|^2 \leqslant \|f\|^2$$

*and*

$$\|v(t)\| \leqslant \frac{1}{\alpha}\|f\|, \forall t \in [-T, T].$$

*Proof.* We have

$$\begin{aligned} \|f\|^2 &= \langle \alpha v(-T) + v(T), \alpha v(-T) + v(T)\rangle \\ &= \alpha^2\|v(-T)\|^2 + \|v(T)\|^2 + 2\alpha \langle v(-T), v(T)\rangle . \end{aligned} \tag{13}$$

Set $h(t) := \langle v(-t), v(t) \rangle, t \in [-T, T]$. We see that

$$h'(t) = 0, \forall t \in (-T, T).$$

Therefore, $h$ is a constant. This implies that $h(0) = h(T)$. Thus, $\langle v(-T), v(T) \rangle = \|v(0)\|^2$.

Set $p(t) := \|v(t)\|^2, \quad t \in [-T, T]$. Then $p'(t) = -2 \langle B(t)v(t), v(t) \rangle \leqslant 0, \quad \forall t \in (-T, T)$. This implies that $p(0) \geqslant p(T)$. Therefore,

$$\langle v(-T), v(T) \rangle = \|v(0)\|^2 \geqslant \|v(T)\|^2.$$

It follows from (13) and the positivity of $\alpha$ that

$$\|f\|^2 \geqslant \alpha^2 \|v(-T)\|^2 + (2\alpha + 1)\|v(T)\|^2.$$

On the other hand, we have $\|v(t)\|^2 = p(t) \leqslant p(-T) = \|v(-T)\|^2, \forall t \in [-T, T]$. Therefore $\|v(t)\| \leqslant \|v(-T)\| \leqslant \dfrac{1}{\alpha}\|f\|, \forall t \in [-T, T]$. The lemma is proved. □

**Lemma 3.4.** *There exists a unique solution of the problem (11).*

*Proof.* Since $B(t)$ $(-T \leqslant t \leqslant T)$ satisfies the assumptions (5),(6) and (8), due to Theorem 3.9 in [12; p. 147], there exists an evolution operator $U(t)$ $(-T \leqslant t \leqslant T)$ which is a bounded linear operator on $H$ such that if $v(t)$ is a solution of the problem $v_t + B(t)v = 0, -T < t < T$, then $v(t) = U(t)v(-T)$.

Let $h(t) = \langle v(-t), v(t) \rangle, \ \forall t \in [-T, T]$. By direct calculation we see that $h'(t) = 0, \forall t \in (-T, T)$. Therefore, $h$ is a constant. This implies that $h(T) = h(0)$. Thus,

$$\begin{aligned} \langle v(-T), v(T) \rangle &= \langle v(0), v(0) \rangle \\ &= \|v(0)\|^2 \geqslant 0. \end{aligned}$$

Therefore,

$$\langle U(T)v(-T), v(-T) \rangle = \langle v(T), v(-T) \rangle = \|v(0)\|^2 \geqslant 0.$$

This implies that the operator $U(T)$ is positive. Therefore the operator $\alpha I + U(T)$ is invertible for all $\alpha > 0$. Finally, set $v(t) = U(t)(\alpha I + U(T))^{-1} f, \ t \in [-T, T]$, by direct calculation, we see that $v(t)$ is a unique solution of the problem (11). □

**Theorem 3.5.** *The following inequality holds for all* $\alpha > 0$

$$2\alpha \left( \|z(0)\| - \frac{E}{2} \right)^2 + \|z(T)\|^2 \leqslant \varepsilon^2 + \frac{\alpha E^2}{2}. \tag{14}$$

*Proof.* Let $q(t) = \langle v(-t), z(t) \rangle, \ \forall t \in [0, T]$. By direct calculation we see that $q'(t) = 0, \forall t \in (0, T)$. Therefore, $q$ is a constant. This implies that $q(T) = q(0)$. Thus,

$$\langle v(-T), z(T) \rangle = \langle v(0), z(0) \rangle.$$

We have

$$\alpha v(-T) - z(T) = f - u(T).$$

Therefore, we obtain

$$
\begin{aligned}
\varepsilon^2 + \frac{\alpha E^2}{2} &\geqslant \|f - u(T)\|^2 + \frac{\alpha E^2}{2} \\
&= \|\alpha v(-T) - z(T)\|^2 + \frac{\alpha E^2}{2} \\
&= \alpha^2 \|v(-T)\|^2 - 2\alpha \langle v(-T), z(T)\rangle + \|z(T)\|^2 + \frac{\alpha E^2}{2} \\
&= \alpha^2 \|v(-T)\|^2 - 2\alpha \langle v(0), z(0)\rangle + \|z(T)\|^2 + \frac{\alpha E^2}{2} \\
&= \alpha^2 \|v(-T)\|^2 + 2\alpha \langle z(0) - u(0), z(0)\rangle + \|z(T)\|^2 + \frac{\alpha E^2}{2} \\
&= \alpha^2 \|v(-T)\|^2 + 2\alpha \|z(0)\|^2 - 2\alpha \langle u(0), z(0)\rangle + \|z(T)\|^2 + \frac{\alpha E^2}{2} \\
&\geqslant 2\alpha \|z(0)\|^2 - 2\alpha \langle u(0), z(0)\rangle + \|z(T)\|^2 + \frac{\alpha E^2}{2} \\
&\geqslant 2\alpha \|z(0)\|^2 - 2\alpha \|u(0)\|\|z(0)\| + \|z(T)\|^2 + \frac{\alpha E^2}{2} \\
&\geqslant 2\alpha \|z(0)\|^2 - 2\alpha E\|z(0)\| + \|z(T)\|^2 + \frac{\alpha E^2}{2} \\
&= 2\alpha \left( \|z(0)\| - \frac{E}{2} \right)^2 + \|z(T)\|^2.
\end{aligned}
$$

The theorem is proved. □

## 3.1 A priori parameter choice rule

**Theorem 3.6.** *Suppose that $u(t)$ is a solution of the problem (1) subjects to the constraint (9), and $v(t)$ is the solution of the problem (11). Then by choosing $\alpha = a\left(\frac{\varepsilon}{E}\right)^2$, $(a > 0)$, we obtain, for all $t \in [0, T]$,*

$$
\|u(t) - v(t)\| \le e^{kt - kT\nu(t)} \sqrt{1 + \frac{a}{2}}^{\nu(t)} \left( \frac{1}{2} + \sqrt{\frac{1}{2}\left(\frac{1}{2} + \frac{1}{a}\right)} \right)^{1-\nu(t)} \varepsilon^{\nu(t)} E^{1-\nu(t)},
$$

*where $\nu(t)$ is defined by (3). In the case of $a = 1$, we have*

$$
\|u(t) - v(t)\| \le \frac{3}{2} e^{kt - kT\nu(t)} \varepsilon^{\nu(t)} E^{1-\nu(t)}, \ \forall t \in [0, T].
$$

*Proof.* Using (4), we obtain

$$
\|z(t)\| \le e^{kt - kT\nu(t)} \|z(T)\|^{\nu(t)} \|z(0)\|^{1-\nu(t)}. \tag{15}
$$

On the other hand, from (14) we have

$$\begin{aligned}\|z(T)\|^2 &\leqslant 2\alpha\left(\|z(0)\| - \frac{E}{2}\right)^2 + \|z(T)\|^2\\ &\leqslant \varepsilon^2 + \frac{\alpha E^2}{2}\\ &= \left(1 + \frac{a}{2}\right)\varepsilon^2\end{aligned}$$

or

$$\|z(T)\| \leqslant \varepsilon\sqrt{1 + \frac{a}{2}}. \tag{16}$$

Furthermore, we have

$$\begin{aligned}2\alpha\left(\|z(0)\| - \frac{E}{2}\right)^2 &\leqslant 2\alpha\left(\|z(0)\| - \frac{E}{2}\right)^2 + \|z(T)\|^2\\ &\leqslant \varepsilon^2 + \frac{\alpha E^2}{2}\\ &= \frac{\alpha E^2}{a} + \frac{\alpha E^2}{2}.\end{aligned}$$

This implies that

$$\left(\|z(0)\| - \frac{E}{2}\right)^2 \leqslant \frac{1}{2}\left(\frac{1}{2} + \frac{1}{a}\right)E^2.$$

Therefore

$$\|z(0)\| - \frac{E}{2} \leqslant E\sqrt{\frac{1}{2}\left(\frac{1}{2} + \frac{1}{a}\right)}$$

or

$$\|z(0)\| \leqslant \left(\frac{1}{2} + \sqrt{\frac{1}{2}\left(\frac{1}{2} + \frac{1}{a}\right)}\right)E. \tag{17}$$

The proposition of Theorem 3.6 follows immediately from (15), (16) and (17). □

### 3.2 A posteriori parameter choice rule

In this section, we denote by $v_\alpha(t)$ the solution of the problem (11).

**Theorem 3.7.** *Suppose that $\varepsilon < \|f\|$. Then there exists a unique number $\alpha_\varepsilon > 0$ such that*

$$\|v_{\alpha_\varepsilon}(T) - f\| = \varepsilon. \tag{18}$$

*Further, if $u(t)$ is a solution of the problem (1) satisfying (9), then*

$$\|u(t) - v_{\alpha_\epsilon}(t)\| \leqslant 2^{\nu(t)}e^{kt-kT\nu(t)}\varepsilon^{\nu(t)}E^{1-\nu(t)},\ \forall t \in [0, T]. \tag{19}$$

*Proof.* Let $\rho(\alpha) = \|v_\alpha(T) - f\| = \alpha\|v_\alpha(-T)\|,\ \forall\alpha > 0$. By similar argument as in [4], we conclude that $\rho$ is a continuous function, $\lim\limits_{\alpha\to 0^+} \rho(\alpha) = 0$, $\lim\limits_{\alpha\to+\infty} \rho(\alpha) = \|f\|$, and $\rho$ is a strictly increasing function. This implies that there exists a unique number $\alpha_\varepsilon > 0$ which satisfies (18).

We now establish error estimate of this method. Let $z(t) = u(t) - v_{\alpha_\varepsilon}(t),\ t \in [0, T]$. We have

$$\begin{aligned}\|z(T)\| &= \|u(T) - v_{\alpha_\varepsilon}(T)\| = \|(u(T) - f) - (v_{\alpha_\varepsilon}(T) - f)\| \\ &\leqslant \|u(T) - f\| + \|v_{\alpha_\varepsilon}(T) - f\| \leqslant 2\varepsilon. \end{aligned} \tag{20}$$

Put $g_{\alpha_\varepsilon} = v_{\alpha_\varepsilon}(-T)$. We have

$$\alpha_\varepsilon g_{\alpha_\varepsilon} + v_{\alpha_\varepsilon}(T) = f$$

and

$$\begin{aligned}\langle g_{\alpha_\varepsilon}, z(T)\rangle &= \langle v_{\alpha_\varepsilon}(0), z(0)\rangle \\ &= \langle u(0) - z(0), z(0)\rangle \\ &= \langle u(0), z(0)\rangle - \|z(0)\|^2.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\varepsilon^2 + \frac{\alpha_\varepsilon E^2}{2} &\geqslant \|f - u(T)\|^2 + \frac{\alpha_\varepsilon E^2}{2} \\ &= \|\alpha_\varepsilon g_{\alpha_\varepsilon} - z(T)\|^2 + \frac{\alpha_\varepsilon E^2}{2} \\ &= \alpha_\varepsilon^2\|g_{\alpha_\varepsilon}\|^2 - 2\alpha_\varepsilon \langle g, z(T)\rangle + \|z(T)\|^2 + \frac{\alpha_\varepsilon E^2}{2} \\ &= \rho^2(\alpha_\varepsilon) - 2\alpha_\varepsilon \left(\langle u(0), z(0)\rangle - \|z(0)\|^2\right) + \|z(T)\|^2 + \frac{\alpha_\varepsilon E^2}{2} \\ &= \varepsilon^2 + 2\alpha_\varepsilon\|z(0)\|^2 - 2\alpha_\varepsilon \langle u(0), z(0)\rangle + \|z(T)\|^2 + \frac{\alpha_\varepsilon E^2}{2} \\ &\geqslant 2\alpha_\varepsilon\|z(0)\|^2 - 2\alpha_\varepsilon \langle u(0), z(0)\rangle + \frac{\alpha_\varepsilon E^2}{2} + \varepsilon^2 \\ &\geqslant 2\alpha_\varepsilon\|z(0)\|^2 - 2\alpha_\varepsilon\|u(0)\|\|z(0)\| + \frac{\alpha_\varepsilon E^2}{2} + \varepsilon^2 \\ &\geqslant 2\alpha_\varepsilon \left(\|z(0)\| - \frac{E}{2}\right)^2 + \varepsilon^2.\end{aligned}$$

This implies that

$$2\alpha_\varepsilon \left(\|z(0)\| - \frac{E}{2}\right)^2 \leqslant \frac{\alpha_\varepsilon E^2}{2}$$

or

$$\|z(0)\| \leqslant E. \tag{21}$$

From (4), (20) and (21), we have

$$\begin{aligned}\|u(t) - v_{\alpha_\varepsilon}(t)\| = \|z(t)\| &\leqslant e^{kt-kT\nu(t)}\|z(T)\|^{\nu(t)}\|z(0)\|^{1-\nu(t)}\\ &\leqslant e^{kt-kT\nu(t)}(2\varepsilon)^{\nu(t)}E^{1-\nu(t)}\\ &= 2^{\nu(t)}e^{kt-kT\nu(t)}\varepsilon^{\nu(t)}E^{1-\nu(t)},\ \forall t \in [0,T].\end{aligned}$$

The theorem is proved. □

## REFERENCES


[1] D. N. Hào, N. V. Duc and H. Sahli, "A non-local boundary value problem method for parabolic equations backward in time," *J. Math. Anal. Appl*, 345, pp. 805-815, 2008.

[2] D. N. Hào, N. V. Duc and D. Lesnic, "A non-local boundary value problem method for the Cauchy problem for elliptic equations," *Inverse Problems*, 25, p. 27, 2009,

[3] D. N. Hào, N. V. Duc and D. "Lesnic, Regularization of parabolic equations backwards in time by a non-local boundary value problem method," *IMA Journal of Applied Mathematics*, 75, pp. 291-315, 2010.

[4] D. N. Hào and N. V. Duc, "Stability results for backward parabolic equations with time dependent coefficients," *Inverse Problems*, Vol. 27, No. 2, 2011.

[5] D. N. Hào and N. V. Duc, "Regularization of backward parabolic equations in Banach spaces," *J. Inverse Ill-Posed Probl*, 20, no. 5-6, pp. 745-763, 2012.

[6] D. N. Hào and N. V. Duc, "A non-local boundary value problem method for semi-linear parabolic equations backward in time," Applicable Analysis, 94, pp. 446-463, 2015.

[7] F. John, "Continuous dependence on data for solutions of partial differential equations with a presribed bound," *Comm. Pure Appl. Math.,* 13, pp. 551-585, 1960.

[8] M. M. Lavrent'ev, V. G. Romanov and G. P. Shishatskii, *Ill-posed Problems in Mathematical Physics and Analysis*, Amer. Math. Soc., Providence, R. I., 1986.

[9] L. Payne, *Improperly Posed Problems in Partial Differential Equations*, SIAM, Philadelphia, 1975.

[10] H. Tanabe, *Equations of Evolution,* Pitman, London, 1979.

[11] A. Tikhonov and V. Y. Arsenin, *Solutions of Ill-posed Problems,* Winston, Washington, 1977.

[12] A. Yagi, *Abstract Parabolic Evolution Equations and their Applications,* SpringerVerlag, Heidelberg, Berlin, 2010.

# TÓM TẮT

## MỘT PHƯƠNG PHÁP CHỈNH HÓA CHO PHƯƠNG TRÌNH PARABOLIC VỚI HỆ SỐ PHỤ THUỘC THỜI GIAN

Cho $H$ là không gian Hilbert với chuẩn $\|\cdot\|$ và $A(t), \quad (0 \leqslant t \leqslant T)$ là toán tử không bị chặn xác định dương từ $D(A(t)) \subset H$ vào $H$. Trong bài báo này, chúng tôi đề xuất một phương pháp chỉnh hóa cho phương trình parabolic ngược thời gian với hệ số phụ thuộc thời gian

$$\begin{cases} u_t + A(t)u = 0, & 0 < t < T, \\ \|u(T) - f\| \leqslant \varepsilon, & f \in H, \varepsilon > 0. \end{cases}$$

Các luật chọn tham số tiên nghiệm và hậu nghiệm được đề xuất kéo theo các đánh giá sai số kiểu Hölder type. Các đánh giá sai số này là sự cải tiến một vài kết quả trong bài báo [4].