INVARIANT DIFFERENTIAL OPERATORS ON THE COMPACTIFICATION OF SYMMETRIC SPACES

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Abstract: Let G be a connected real semisimple Lie group with finite center and θ be a Cartan involution of G. Suppose that K is the maximal compact subgroup of G corresponding to the Cartan involution θ. The coset space $\mathbf{X} = G/K$ is then a Riemannian symmetric space. Denote by $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$ into eigenspaces of θ. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$ and Σ be the corresponding restricted root system. In [5], by choosing $\Sigma' = \{\alpha \in \Sigma \mid 2\alpha \notin \Sigma;\ \frac{\alpha}{2} \notin \Sigma\}$ instead of the restricted root system Σ and using the action of the Weyl group, we constructed a compact real analytic manifold $\widehat{\mathbf{X}}'$ in which the Riemannian symmetric space G/K is realized as an open subset and that G acts analytically on it. In our construction, the real analytic structure of $\widehat{\mathbf{X}}'$ induced from the real analytic srtucture of $\widehat{A}_{\mathbb{R}}$, the compactification of the vectorial part. The purpose of this note is to show that the system of invariant differential operators on $\mathbf{X} = G/K$ can extends analytically on $\widehat{\mathbf{X}}'$.
Keywords: Symmetric spaces, Weyl group, Cartan decomposition, compactification.

1 INTRODUCTION

Let G be a connected real semisimple Lie group with finite center and $\mathfrak{g}$ be the Lie algebra of G. Denote by θ the Cartan involution of G and K the fixed points of θ. Then K is a maximal compact subgroup of G and the coset space $\mathbf{X} = G/K$ becomes a Riemannian symmetric space. We also denote by θ the Cartan involution of $\mathfrak{g}$ corresponding to the Cartan involution θ of G. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ into eigenspaces of θ, where $\mathfrak{k}$ is the Lie algebra of K.

Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$ and $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$. The corresponding analytic subgroup A of $\mathfrak{a}$ in G is then called the vectorial part of X. For a non zero $\alpha \in \mathfrak{a}^*$, the non zero eigenspace

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \mid [H, Y] = \alpha(H)Y,\ \forall H \in \mathfrak{a}\}$$

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is called the root space and the corresponding $\alpha's$ the restricted root. Then the set $\Sigma = \{\alpha \in \mathfrak{a}^* \mid \mathfrak{g}_\alpha \neq \{0\}, \alpha \neq 0\}$ defines a root system with the inner product induced by the Killing form $<,>$ of $\mathfrak{g}$. Moreover, the Weyl group W of Σ is defined with the normalizer $N_K(\mathfrak{a})$ of $\mathfrak{a}$ in K modulo the centralizer $M = Z_K(\mathfrak{a})$ of $\mathfrak{a}$ in K. It acts naturally on $\mathfrak{a}$ and coincides via this action with the reflection group of Σ.

Choose a fundamental system $\Delta = \{ \alpha_1, ..., \alpha_l \}$ of Σ, where the number l which equals dim $\mathfrak{a}$ is called the split rank of the symmetric space X and denote Σ^+ the corresponding set of all restricted positive roots in Σ.

Denote by $\mathfrak{g}_{\mathbf{C}}$ the complexification of $\mathfrak{g}$ and $G_{\mathbf{C}}$ the corresponding analytic group. Let $\mathfrak{a}_{\mathbf{C}}$ be the complexification of $\mathfrak{a}$ and $A_{\mathbf{C}}$ be the analytic subgroup of $\mathfrak{a}_{\mathbf{C}}$ in $G_{\mathbf{C}}$. For each $a \in A_{\mathbf{C}}$ and $\alpha \in \Sigma$ we define $a^\alpha = e^{\alpha log\ a} \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$ and consider the subset

$$A_{\mathbb{R}} = \{ a \in A_{\mathbf{C}} \mid a^\alpha \in \mathbb{R},\ \forall \alpha \in \Sigma \}.$$

Let $(\mathbf{C}^*)^\Sigma$ be the set of complexes $z = (z_\beta)_{\beta \in \Sigma}$, where $z_\beta \in \mathbf{C}^*$ and $\mathbf{C}\mathbb{P}^1$ be the 1-dimensional complex projective space. Then we can define a map

$$\varphi : A_{\mathbf{C}} \longrightarrow (\mathbf{C}^*)^\Sigma,\ a \mapsto \varphi(a) = (a^\alpha)_{\alpha \in \Sigma}.$$

In [2], based on the natural imbedding of $(\mathbf{C}^*)^\Sigma$ into $(\mathbf{C}\mathbb{P}^1)^\Sigma$, we constructed an imbedding of $A_{\mathbb{R}}$ into a compact real analytic manifold $\widehat{A}_{\mathbb{R}}$ which is called a compactification of $A_{\mathbb{R}}$.

In [5], by choosing the reduced root system $\Sigma' = \{\alpha \in \Sigma \mid 2\alpha \notin \Sigma;\ \frac{\alpha}{2} \notin \Sigma\}$ instead of Σ and using the action of the Weyl group, we constructed a compact real analytic manifold $\widehat{\mathbf{X}}'$ in which the Riemannian symmetric space G/K is realized as an open subset and that G acts analytically on it. Moreover, the real analytic structure of $\widehat{\mathbf{X}}'$ induced from the real analytic srtucture of $\widehat{A}_{\mathbb{R}}$. Our construction is a motivation of the construction of T. Oshima and J. Sekiguchi [9] for affine symmetric spaces and it is similar to those in N. Shimeno [10] for semismple symmetric spaces.

In this note, first we recall some notation and results concerning the compactification of Riemannian symmetric spaces constructed in [5] and then we show that the system of invariant differential operators on $\mathbf{X} = G/K$ can extends analytically on the compactification $\widehat{\mathbf{X}}'$.

2 A REALIZATION OF RIEMANNIAN SYMMETRIC SPACES

In this section, we recall some notation and results concerning the compactification of Riemannian symmetric spaces constructed in [5].

Let G be a connected real semisimple Lie group with finite center and $\mathfrak{g}$ be the Lie algebra of G. Denote by $\mathfrak{g}_{\mathbf{C}}$ the complexification of $\mathfrak{g}$ and $G_{\mathbf{C}}$ the corresponding analytic group. For simplicity, we assume that G is the real form of the complex Lie group $G_{\mathbf{C}}$. Let $\mathfrak{a}_{\mathbf{C}}$ be the complexification of $\mathfrak{a}$ and $A_{\mathbf{C}}$ be the analytic subgroup of $\mathfrak{a}_{\mathbf{C}}$ in $G_{\mathbf{C}}$. Then we can consider the map $\varphi : A_{\mathbf{C}} \longrightarrow (\mathbf{C}^*)^{\Sigma}$ which is defined by $\varphi(a) = (a^{\alpha})_{\alpha\in\Sigma}$, $\forall a \in A_{\mathbf{C}}$, where $(\mathbf{C}^*)^{\Sigma}$ is the set of complexes $z = (z_{\beta})_{\beta\in\Sigma}$.

It follows that for every $z = (z_{\alpha})_{\alpha\in\Sigma} \in \varphi(A_{\mathbf{C}})$ we have

$$z_{-\alpha} = (z_{\alpha})^{-1}, \ \forall\alpha \in \Sigma \tag{2.1}$$

$$z_{\alpha} = \prod_{\gamma\in\Delta} (z_{\gamma})^{k(\alpha,\gamma)}, \ \forall\alpha \in \Sigma^{+}, \ \alpha = \sum_{\gamma\in\Delta} k(\alpha,\gamma).\gamma. \tag{2.2}$$

Denote $\mathbf{C}\mathbb{P}^1$ the 1-dimensional complex projective space. Then, based on the natural imbedding of $(\mathbf{C}^*)^{\Sigma}$ into $(\mathbf{C}\mathbb{P}^1)^{\Sigma}$, we get an imbedding map of $A_{\mathbf{C}}$ into $(\mathbf{C}\mathbb{P}^1)^{\Sigma}$ denoted also by φ. Now for each $a \in A_{\mathbf{C}}$ and $\alpha \in \Sigma$ we define $a^{\alpha} = e^{\alpha log\, a} \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$ and consider the subset

$$A_{\mathbb{R}} = \{ \ a \in A_{\mathbf{C}} \mid a^{\alpha} \in \mathbb{R}, \ \forall\alpha \in \Sigma \ \}.$$

By definition, $\varphi(A_{\mathbb{R}})$ is a subset of $(\mathbb{R}\mathbb{P}^1)^{\Sigma}$. Let $\widehat{A}_{\mathbb{R}}$ be the closure of $\varphi(A_{\mathbb{R}})$ in $(\mathbb{R}\mathbb{P}^1)^{\Sigma}$. Denote by $\mathcal{U}_{\Delta}$ the subset of $\widehat{A}_{\mathbb{R}}$ consists of elements $m = (m_{\alpha}, m_{-\alpha})$, for all $\alpha \in \Sigma^{+}$ such that

$$m_{\alpha} = \prod_{\gamma\in\Delta} (m_{\gamma})^{k(\alpha,\gamma)}, \ \alpha = \sum_{\gamma\in\Delta} k(\alpha,\gamma).\gamma$$

and $m_{-\alpha} = m_{\alpha}^{-1}$.

Then $\mathcal{U}_{\Delta}$ is an open subset in $\widehat{A}_{\mathbb{R}}$ and we get a homeomorphism $\chi_{\Delta} : \mathcal{U}_{\Delta} \longrightarrow \mathbb{R}^{\Delta}$ defined by $\chi_{\Delta}(m) = (m_{\gamma})_{\gamma\in\Delta}$, $\forall m \in \mathcal{U}_{\Delta}$. Moreover, it follows from [2, Theorem 1.4] that $\widehat{A}_{\mathbb{R}}$ is a compact real analytic manifold that is called a compactification of $A_{\mathbb{R}}$ and the set of charts $\{(\mathcal{U}_{w(\Delta)}, \chi_{w(\Delta)})\}_{w\in W}$ defines an atlas of charts on $\widehat{A}_{\mathbb{R}}$ so that the manifold $\widehat{A}_{\mathbb{R}}$ is covered by $|W|$-many charts.

Consider the subset $\widehat{A}_{\mathbb{R}}^{-} = \{ \ \tilde{a} \in \widehat{A}_{\mathbb{R}} \mid (\tilde{a})^{\alpha} \in [-1,1], \forall\alpha \in \Sigma \ \}$ and recall that the Weyl group W acts on $\widehat{A}_{\mathbb{R}}$ by $(w.\tilde{a})_{\alpha} = (\tilde{a})_{w^{-1}\alpha}$, $\forall w \in W$, $\forall\tilde{a} \in \widehat{A}_{\mathbb{R}}$. Since $A_{\mathbb{R}}$ acts naturally on $\widehat{A}_{\mathbb{R}}$, we see that for each $\tilde{a} \in \widehat{A}_{\mathbb{R}}^{-}$, there exists $t \in [-1,1]^{\Delta}$ and $a_t \in A_{\mathbb{R}}$ such that $\tilde{a} = a_t.sgn\ t$ and this decomposition is unique. Here $sgn\ t = (sgn\ t_{\gamma})_{\gamma\in\Delta}$ and for an s in $\mathbb{R}$ we define $sgn\ s = 1$ (resp. $0, -1$) if $s > 0$ (resp. $s = 0, s < 0$). Moreover, by choosing a suitable positive system Σ^{+} we obtain $W.\widehat{A}_{\mathbb{R}}^{-} = \widehat{A}_{\mathbb{R}}$.

Note that for $\tilde{a} \in \widehat{A}_{\mathbb{R}}^{-}$, we obtain $\epsilon(\tilde{a}) \in \{-1,0,+1\}^{\Delta}$ and for all $\alpha \in \Sigma$ so that $\alpha = \sum_{\gamma\in\Delta} k(\alpha,\gamma).\gamma$, we have

$$\epsilon(\tilde{a})^{\alpha} = \prod_{\gamma\in\Delta} (\epsilon(\tilde{a})^{\gamma})^{|k(\alpha,\gamma)|}.$$

It follows that the mapping ϵ of Σ to $\{-1,0,+1\}$ defined by

$$\epsilon : \Sigma \longrightarrow \{-1,0,+1\}, \ \alpha \mapsto \epsilon(\tilde{a}) = \epsilon(\tilde{a})^{\alpha}$$

is an extended signature of roots that is defined in [9, Definition 2.1].

Now we go to define parabolic subalgebras with respect to extended signatures of roots $\epsilon = \epsilon(\tilde{a})$, for all $\tilde{a} \in \widehat{A}_{\mathbb{R}}$.

First we consider $\tilde{a} \in \widehat{A}_{\mathbb{R}}^{-}$ and denote $\epsilon = \epsilon(\tilde{a})$ the corresponding extended signature of roots. Put $F_\epsilon = \{ \gamma \in \Delta \mid \epsilon(\gamma) = \epsilon(\tilde{a})^\gamma \neq 0 \}$ and $\Sigma_{F_\epsilon} = (\sum_{\gamma \in F_\epsilon} \mathbb{R}\gamma) \cap \Sigma$.

Then as in [9], we can define a parabolic subalgebra $\mathfrak{p}_\epsilon$ in $\mathfrak{g}$ with $\mathfrak{p}_\epsilon = \mathfrak{m}_\epsilon + \mathfrak{a}_\epsilon + \mathfrak{n}_\epsilon$ is the corresponding Langlands decomposition. Denote P_ϵ the parabolic subgroup in G with respect to $\mathfrak{p}_\epsilon$, we see that $P_\epsilon = M_\epsilon A_\epsilon N_\epsilon$ is the corresponding Langlands decomposition of P_ϵ, where $A_\epsilon,\ N_\epsilon,\ (M_\epsilon)_0$ are the analytic subgroups of G, respectively, to $\mathfrak{a}_\epsilon,\ \mathfrak{n}_\epsilon,\ \mathfrak{m}_\epsilon$ and $M_\epsilon = (M_\epsilon)_0 M$.

Moreover, it follows from [9, Lemma 2.3] that $P(\epsilon) = (M_\epsilon \cap K) A_\epsilon N_\epsilon$ is a closed subgroup of G and the map

$$N^- \times A(\epsilon) \times P(\epsilon) \longrightarrow G,\ (n, a, p) \mapsto nap$$

is an analytic diffeomorphism onto an open submanifold of G.

In general, for each $\tilde{\eta} = w.\tilde{a} \in \widehat{A}_{\mathbb{R}}$, where $w \in W$ and $\tilde{a} \in \widehat{A}_{\mathbb{R}}^{-}$, we firstly consider the parabolic subgroup $P_\epsilon = M_\epsilon A_\epsilon N_\epsilon$ with respect to $\epsilon = \epsilon(\tilde{a})$, the corresponding extended signature of $\tilde{a}$. Then we can define a parabolic subgroup $P_{\tilde{\eta}} = \underline{w}.P_\epsilon.\underline{w}^{-1}$ based on the action of the Weyl group W on the parabolic subgroup P_ϵ. Here $\underline{w}$ denote a representative for $w \in W$ in $N_K(\mathfrak{a})$ (see [1]).

Now we put $\Sigma' = \{\alpha \in \Sigma \mid 2\alpha \notin \Sigma;\ \frac{\alpha}{2} \notin \Sigma\}$ and denote $\Sigma'_\epsilon = \{\alpha \in \Sigma' \mid \epsilon(\alpha) = 1\}$ for every extended signature ϵ of roots defined by $\epsilon(\tilde{a})$. Then (see [9]) it follows that Σ' and Σ'_ϵ are reduced root systems. Let $W',\ W'_\epsilon$ and W'_{F_ϵ} be the subgroups of W generated by the reflections with respect to the roots in $\Sigma',\ \Sigma'_\epsilon$ and Σ'_{F_ϵ}.

Denote $\widehat{A}'_{\mathbb{R}} = W'.\widehat{A}_{\mathbb{R}}^{-}$ and consider the product manifold $G \times \widehat{A}'_{\mathbb{R}}$. Let $x = (g, \tilde{\eta})$ be an element of $G \times \widehat{A}_{\mathbb{R}}$, where $\tilde{\eta} = w.\tilde{a}$, in which $w \in W'$ and $\tilde{a} \in \widehat{A}_{\mathbb{R}}^{-}$. Then the extended signature of roots with respect to $\tilde{a}$ denoted by $\epsilon_x = \epsilon(\tilde{a})$. For simplicity, we denote $P(x),\ F_x,\ \Sigma_x,\ \Sigma'_x,\ W'_x, ...$ instead of $P(\epsilon_x),\ F_{\epsilon_x},\ \Sigma_{\epsilon_x},\ \Sigma'_{\epsilon_x},\ W'_{F_{\epsilon_x}}, ...$, respectively.

Let $\{H_1, H_2, ..., H_l\}$ denote the dual basis of $\Delta = \{\alpha_1, ..., \alpha_l\}$, that is, $H_j \in \mathfrak{a}$ and $\alpha_i(H_j) = \delta_{ij},\ \forall i, j = 1, 2, ..., l$. Put $a(x) = exp(-\frac{1}{2} \sum_{\gamma \in F_x} log|t_\gamma|\ H_\gamma)$, where H_γ is in $\{H_1, H_2, ..., H_l\}$ with respect to γ and denote $W(x) = \{w \in W_x \mid \Sigma_x \cap w\Sigma^+ = \Sigma_x \cap \Sigma^+\}$.

Definition 2.1. We say that two elements $x = (g, \omega.\tilde{a})$ and $x' = (g', \omega'.\tilde{a})$ of $G \times \widehat{A}'_{\mathbb{R}}$ are equivalent if and only if the following conditions hold:

(i) $w.\epsilon_x = w'.\epsilon'_x$

(ii) $w^{-1}w' \in W(x)$

(iii) $ga(x)P(x)\underline{w} = g'a(x')P(x)\underline{w}'$.

Then it follows that (see [9]) Definition 2.1 really gives an equivalence relation, which we write $x \sim x'$. Moreover, we see that the action of G on $G \times \widehat{A}'_{\mathbb{R}}$ are compatible with the equivalence relation and the quotient space of $G \times \widehat{A}'_{\mathbb{R}}$ by this equivalence relation then becomes a topological space with the quotient topology and denoted by $\widehat{\mathbf{X}}'$.

Let $\pi : G \times \widehat{A}'_{\mathbb{R}} \longrightarrow \widehat{\mathbf{X}}'$ be the natural projection. Since the action of G on $G \times \widehat{A}'_{\mathbb{R}}$ are compatible with the equivalence relation, we can define an action of G on $\widehat{\mathbf{X}}'$ by

$$g_1\pi(g, \tilde{a}) = \pi(g_1 g, \tilde{a}),\ \forall g, g_1 \in G, \tilde{a} \in \widehat{A}'_{\mathbb{R}}. \tag{2.3}$$

Now consider the atlas of charts $\{(\mathcal{U}_{w(\Delta)}, \chi_{w(\Delta)})\}_{w\in W}$ on $\widehat{A}_{\mathbb{R}}$ defined in [2, Theorem 1.4], where $\mathcal{U}_{w(\Delta)} = w.\mathcal{U}_{\Delta}$ and $\chi_{w(\Delta)} : \mathcal{U}_{w(\Delta)} \longrightarrow \mathbb{R}^{w(\Delta)}$ is a homeomorphism defined by

$$\chi_{w(\Delta)}(w.m) = (m_{w^{-1}.\gamma})_{\gamma\in\Delta},\ \forall m \in \mathcal{U}_{\Delta}, w \in W.$$

For every $g \in G$ and $w \in W'$, we put $\Omega_g^w = \pi(gN^- \times \mathcal{U}_{w(\Delta)})$, in which N^- is the analytic subgroup of G corresponding to $\mathfrak{n}^- = \theta(\mathfrak{n})$, where $\mathfrak{n} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ and define a map

$$\Phi_g^w : N^- \times \mathbb{R}^\Delta \longrightarrow \Omega_g^w$$

by $\Phi_g^w(n, t) = \pi(gn, w.\tilde{a}_t),\ \forall (n, t) \in N^- \times \mathbb{R}^\Delta$.

Based on this, we get the following theorem [5, Theorem 3.5].

Theorem 2.2. *The topological space $\widehat{\mathbf{X}}'$ have the following properties:*

(i) $\widehat{\mathbf{X}}'$ is a compact connected real analytic manifold and $\bigcup_{w\in W', g\in G} \Omega_g^w$ is an open covering of $\widehat{\mathbf{X}}'$ such that the maps Φ_g^w are real analytic diffeomorphisms.

(ii) The action of G on $\widehat{\mathbf{X}}'$ is analytic and the orbit $G\pi(x)$ for a point x in $\widehat{\mathbf{X}}'$ is isomorphic to the homogeneous space $G/P(x)$. In particular, the number of G-orbits which are isomorphic to G/K (resp. G/P) are just the number of elements in W'.

3 INVARIANT DIFFERENTIAL OPERATORS

Let G be a connected real semisimple Lie group with finite center and θ be a Cartan involution of G. Suppose that K is the maximal compact subgroup of G corresponding to the Cartan involution θ. The coset space $\mathbf{X} = G/K$ is then a Riemannian symmetric space. In [2], by choosing $\Sigma' = \{\alpha \in \Sigma \mid 2\alpha \notin \Sigma;\ \frac{\alpha}{2} \notin \Sigma\}$ instead of the restricted root system Σ and using the action of the Weyl group, we constructed a compact real analytic manifold $\widehat{\mathbf{X}}'$ in which the Riemannian symmetric space G/K is realized as an open subset and that G acts analytically on it. In this section, we shall show that the system of invariant differential operators on the symmetric space $\mathbf{X} = G/K$ can extends analytically on the compact G-space $\widehat{\mathbf{X}}'$.

First we recall after [7] on the structure of the algebra of invariant differential operators on G/K. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping of $\mathfrak{g}_{\mathbf{C}}$, which is naturally identified with $\mathbf{D}(G)$, the totality of the left G-invariant differential operators on G. Denote by $\mathbf{D}(G/K)$ the algebra of left G-invariant differential operators on G/K and put

$$\mathcal{U}(\mathfrak{g})^K = \{D \in \mathcal{U}(\mathfrak{g}) \mid Ad(k)D = D \text{ for any } k \in K\}.$$

Then $\mathbf{D}(G/K)$ is a polynomial ring over $\mathbf{C}$ and there exists a natural homomorphism of $\mathcal{U}(\mathfrak{g})^K$ onto $\mathbf{D}(G/K)$ with the kernel $\mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathfrak{k}$.

For a Lie subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, let denote $\mathcal{U}(\mathfrak{b})$ the universal enveloping algebra of $\mathfrak{b}_{\mathbf{C}}$. Then we can naturally identify $\mathcal{U}(\mathfrak{b})$ with a subalgebra of $\mathcal{U}(\mathfrak{g})$. Let $\tilde{\xi}$ be the natural surjective homomorphism of $\mathcal{U}(\mathfrak{g})^K$ onto $\mathbf{D}(G/K)$ with the kernel $\mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathfrak{k}$. It follows that there is an isomorphism ξ between $\mathbf{D}(G/K)$ and $\mathcal{U}(\mathfrak{g})^K/(\mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathfrak{k})$. Moreover, since the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \overline{\mathfrak{n}}$, we see that for any $D \in \mathbf{D}(G/K)^K$ there exists a unique element $D' \in \mathcal{U}(\mathfrak{a} + \overline{\mathfrak{n}})$ such that $D' - D \in \mathcal{U}(\mathfrak{g})\mathfrak{k}$.

Now we review the structure of invariant differential operators on G/K. First the complex linear extension of the involution θ on $\mathfrak{g}_{\mathbf{C}}$ is also denoted by the same letters. Denote by $\Sigma(\mathfrak{b})$ the root system for the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{a}_{\mathbf{C}})$ and $\Sigma(\mathfrak{a})^+$ the set of positive roots with respect to a compatible orders for $\theta(\mathfrak{a})$ and θ. Put $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma(\mathfrak{a})^+} \alpha$. Denote by $\mathfrak{n}_{\mathbf{C}}$ the nilpotent subalgebra of $\mathfrak{g}_{\mathbf{C}}$ corresponding to $\theta(\mathfrak{a})^+$ and $\overline{\mathfrak{n}}_{\mathbf{C}} = \theta(\mathfrak{n}_{\mathbf{C}})$. From the Iwasawa decomposition $\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \overline{\mathfrak{n}}_{\mathbf{C}} \oplus \mathfrak{a}_{\mathbf{C}}$ and the Poincare-Birkhoff-Witt theorem, it follows that

$$\mathcal{U}(\mathfrak{g}) = \overline{\mathfrak{n}}_{\mathbf{C}}\, \mathcal{U}(\overline{\mathfrak{n}}_{\mathbf{C}} \oplus \mathfrak{a}_{\mathbf{C}}) \oplus \mathcal{U}(\mathfrak{a}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{h}. \tag{3.1}$$

Then for any $D \in \mathbf{D}(G/K)^K$ there exists a unique element $D'_{\mathfrak{a}} \in \mathcal{U}(\mathfrak{a})$ such that $D'_{\mathfrak{a}} - D \in \overline{\mathfrak{n}}_{\mathbf{C}}\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{k}$. Let denote

$$\mathcal{U}(\mathfrak{a})^W = \{D \in \mathcal{U}(\mathfrak{a}) \mid Ad(\underline{w})D = D \text{ for any } w \in W\}$$

and put $D_{\mathfrak{a}} = e^{\rho} \circ D'_{\mathfrak{a}} \circ e^{-\rho}$, where e^{ρ} is the function on A defined by $e^{\rho}(a) = e^{\rho(loga)}$ for all $a \in \mathfrak{a}$. Then the map

$$\tilde{\mu} : \mathcal{U}(\mathfrak{g})^K \longrightarrow \mathcal{U}(\mathfrak{a}),\ D \mapsto D_{\mathfrak{a}}$$

defines a surjective homomorphism of $\mathcal{U}(\mathfrak{g})^K$ onto $\mathcal{U}(\mathfrak{a})^W$ with the kernel $\mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathfrak{k}$. Hence, based on the isomorphism ξ, we see that $\tilde{\mu}$ induces the algebra isomorphism

$$\mu : \mathbf{D}(G/K) \longrightarrow \mathcal{U}(\mathfrak{a})^W$$

by identifying algebras $\mathbf{D}(G/K)$ and $\mathcal{U}(\mathfrak{g})^K/(\mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\mathfrak{k})$.

Now we will study G-invariant differential operators on the G-manifold $\widehat{\mathbf{X}}'$ constructed in Section 2 based on the invariant differential operators on the manifold $\mathbf{X} = G/K$.

Consider an element $\tilde{a} \in \widehat{A}_{\mathbb{R}}^{-}$ such that $\epsilon(\tilde{a}) \in \{-1,+1\}^{\Delta}$ and denote by $\epsilon = \epsilon(\tilde{a})$ the corresponding (extended) signature of roots. Then, by Definition 1.1 in [9], we can determine an involution θ_ϵ induced from the Cartan involution θ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k}_\epsilon + \mathfrak{p}_\epsilon$ is the decomposition of $\mathfrak{g}$ into eigenspaces $\mathfrak{k}_\epsilon$ and $\mathfrak{p}_\epsilon$ of θ_ϵ, with respect to eigenvalues $+1$ and -1.

Let $(K_\epsilon)_0$ be the analytic subgroup of G corresponding to Lie subalgebra $\mathfrak{k}_\epsilon$ and denote $K_\epsilon = (K_\epsilon)_0 M$. Then, by using Lemma 1.4 (ii) in [9], we see that K_ϵ is a closed subgroup of G with $\mathfrak{k}_\epsilon$ is its Lie algebra and in the case of θ_ϵ extended into an involution of G, denoted also by θ_ϵ, the closed subgroup K_ϵ is θ_ϵ-invariant.

Moreover, the adjoint representation Ad of G induces an isomorphism between the homogenous space G/K_ϵ and the space $Int\mathfrak{g}/Ad(K_\epsilon)$, where $Int\mathfrak{g}$ is the adjoint group of $\mathfrak{g}$. Then it follows from [9, Remark 1.5] that G/K_ϵ becomes a symmetric homogenous space and called an affine symmetric space.

For every $w \in W'$ and $\epsilon \in \{-1,1\}^{\Delta}$, consider $\widehat{A}'_{\mathbb{R},\epsilon} = \{w.\tilde{a}_t \in \widehat{A}'_{\mathbb{R}} \mid \epsilon(\tilde{a}_t) = \epsilon\}$ and denote $\widehat{\mathbf{X}}'_{w,\epsilon} = \pi(G \times \widehat{A}'_{\mathbb{R},\epsilon})$ the corresponding orbit in $\widehat{\mathbf{X}}'$.

Consider the subset $F_\epsilon = \{ \gamma \in \Delta \mid \epsilon_{\tilde{a}}(\gamma) = \epsilon(\tilde{a})^\gamma \neq 0 \}$ of Δ corresponding to the extended signature of roots $\epsilon \in \{-1,0,1\}^{\Delta}$ and denote $P(\epsilon) = (M_\epsilon \cap K) A_\epsilon N_\epsilon$ the closed subgroup of G with respect to the signature ϵ considered in the previous subsection. In the case of $F_\epsilon = \Delta$ for every $\epsilon = \epsilon(\tilde{a})$; that is ϵ becomes a signature of roots, we see that $W_{F_\epsilon} = W$, $M_\epsilon = G$ and $P(\epsilon) = K_\epsilon$.

Then, based on Theorem 2.5, we get the following corollary.

Corollary 3.1. *For every $w \in W'$ and $\epsilon \in \{-1,1\}^{\Delta}$, there exists an isomorphism*

$$\lambda_\epsilon^w : G/K_\epsilon \longrightarrow \widehat{\mathbf{X}}'_{w,\epsilon} \tag{3.2}$$

defined by $\lambda_\epsilon^w(gK_\epsilon) = \pi(g, \omega.\tilde{a}_t)$, for all $g \in G$ and $\epsilon(\tilde{a}_t) = \epsilon$.

Denote $\mathbf{D}(G/K_\epsilon)$ the algebra of G-left invariant differential operators on G/K_ϵ and consider $\mathcal{U}(\mathfrak{g})^{K_\epsilon} = \{D \in \mathcal{U}(\mathfrak{g}) \mid Ad(k)D = D \text{ for any } k \in K_\epsilon\}$. Then, by a similar argument as the case of $\mathbf{D}(G/K)$, there exists a canonical algebra surjective homomorphism $\tilde{\mu}_\epsilon$ of $\mathcal{U}(\mathfrak{g})^{K_\epsilon}$ onto $\mathbf{D}(G/K_\epsilon)$ with its kernel is $\mathcal{U}(\mathfrak{g})^{K_\epsilon} \cap \mathcal{U}(\mathfrak{g})\mathfrak{k}_\epsilon$.

Denote σ_ϵ the automorphism of $\mathfrak{g}_{\mathbf{C}}$ defined in [9, Lemma 1.3] and consider the automorphism of $\mathcal{U}(\mathfrak{g})$ naturally induced by the automorphism σ_ϵ that is also denoted by σ_ϵ. Then, applying Lemma 2.24 in [9], we get $\mathcal{U}(\mathfrak{g})^{K_\epsilon} = \sigma_\epsilon(\mathcal{U}(\mathfrak{g})^K)$. It follows that there is an isomorphism between $\mathcal{U}(\mathfrak{g})^{K_\epsilon}$ and $\mathcal{U}(\mathfrak{g})^K$. Combining this with the algebra isomorphism μ, we see that the surjective homomorphism $\tilde{\mu}_\epsilon$ induces an isomorphism between algebras $\mathbf{D}(G/K_\epsilon)$ and $\mathcal{U}(\mathfrak{a})^W$. In other words, we obtain the following lemma.

Lemma 3.2. *For every signature of root $\epsilon \in \{-1,1\}^{\Delta}$, there exists an isomorphism*

$$\mu_\epsilon : \mathbf{D}(G/K_\epsilon) \longrightarrow \mathcal{U}(\mathfrak{a})^W \tag{3.3}$$

between algebras $\mathbf{D}(G/K_\epsilon)$ and $\mathcal{U}(\mathfrak{a})^W$.

Now, we go to determine G-invariant differential operators on the G-compact manifold $\widehat{\mathbf{X}}'$ based on the invariant differential operators on the affine symmetric space G/K_ϵ. Denote by $\mathbf{D}(\widehat{\mathbf{X}}')$ the algebra of G-invariant differential operators on the manifold $\widehat{\mathbf{X}}'$ whose coefficients are real analytic functions. Then we have the following theorem.

Theorem 3.3. *For every $w \in W'$ and $\epsilon \in \{-1,1\}^\Delta$, there exists an algebra isomorphism*

$$\lambda : \mathbf{D}(\widehat{\mathbf{X}}') \longrightarrow \mathcal{U}(\mathfrak{a})^W$$

that is given by $\lambda(D) = \mu_\epsilon \circ (\lambda_\epsilon^w)^{-1}(D|\widehat{\mathbf{X}}'_{w,\epsilon})$ for all $D \in \mathbf{D}(\widehat{\mathbf{X}}')$, which does not depend on the choice of a signature ϵ of roots in $\{-1,1\}^\Delta$ and an element w in W'.

Chứng minh. Because of $\widehat{\mathbf{X}}'_{w,\epsilon}$ is open in the connected manifold $\widehat{\mathbf{X}}'$ and μ_ϵ is an isomorphsm, it follows that λ is injective. Now we consider an element $D_\mathfrak{a} \in \mathcal{U}(\mathfrak{a})^W$. To get the Theorem we have only to prove the existence of a differential operator D on $\widehat{\mathbf{X}}'$ satisfying

$$\mu_\epsilon \circ (\lambda_\epsilon^w)^{-1}(D|\widehat{\mathbf{X}}'_{w,\epsilon}) = D_\mathfrak{a}.$$

Indeed, we first prove that

$$\lambda_\epsilon^w \circ (\mu_\epsilon)^{-1}(D_\mathfrak{a}) = \lambda_{\epsilon'}^{w'} \circ (\mu_{\epsilon'})^{-1}(D_\mathfrak{a}) \tag{3.4}$$

for a choice of signatures $\epsilon,\ \epsilon'$ of roots in $\{-1,1\}^\Delta$ and elements $w,\ w'$ in W' such that $\widehat{\mathbf{X}}'_{w,\epsilon} = \widehat{\mathbf{X}}'_{w',\epsilon'}$.

Since $\tilde{\mu}$ is a surjective homomorphism of $\mathcal{U}(\mathfrak{g})^K$ onto $\mathcal{U}(\mathfrak{a})^W$, we can choose $\tilde{D} \in \mathcal{U}(\mathfrak{g})^K$ so that $\tilde{\mu}(\tilde{D}) = D_\mathfrak{a}$.
Then, based on the definitions of $\widehat{\mathbf{X}}'$ and μ_ϵ we see that (3.4) is equivalent to

$$\tilde{\mu}_\epsilon \circ \sigma_\epsilon(\tilde{D}) = \tilde{\mu}_\epsilon \circ Ad(\underline{w}^{-1}\underline{w'})\sigma_{\epsilon'}(\tilde{D}) \tag{3.5}$$

if $\widehat{\mathbf{X}}'_{w,\epsilon} = \widehat{\mathbf{X}}'_{w',\epsilon'}$.

Now by the same argument as the proof of Proposition 2.26 in [9], we can prove that for a choice of signatures $\epsilon,\ \epsilon'$ of roots in $\{-1,1\}^\Delta$ and elements $w,\ w'$ in W' such that $\widehat{\mathbf{X}}'_{w,\epsilon} = \widehat{\mathbf{X}}'_{w',\epsilon'}$, the formula (3.5) is true. In other words, the formula (3.4) is true.
Moreover, when ϵ is a trivial signature; that is $\epsilon = 1$, it follows that $\lambda_1^w \circ (\mu_1)^{-1}(D_\mathfrak{a})$ can be analytically extended to a differential operator D_g^w on Ω_g^w such that

$$D_g^w|\Omega_g^w \cap \widehat{\mathbf{X}}'_{w,\epsilon} = \lambda_\epsilon^w \circ (\mu_\epsilon)^{-1}(D_\mathfrak{a})|\Omega_g^w \cap \widehat{\mathbf{X}}'_{w,\epsilon}$$

for every signature ϵ of roots and $w \in W'$.
Based on this and the formula (3.4), there exists a differential operetor D on $\widehat{\mathbf{X}}'$ satisfying $D|\Omega_g^w = D_g^w$ for any $g \in G$ and $w \in W'$.
Then, it follows from the uniqueness of the analytic continuation that the operator D is G-invariant and the theorem follows. □

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REFERENCES

[1] A. Borel and Lizhen Ji, *Compactifications of symmetric spaces I*, Lectures for the European School of Group Theory, Luminy, France, 2001.

[2] Tran Dao Dong and Tran Vui, *A realization of Riemannian symmetric spaces in compact manifolds*, Proc. of the ICAA Bangkok, (2002) 188-196.

[3] Tran Dao Dong and Tran Vui, *A Compact Imbedding of semi simple symmetric spaces*, East West Journal, 01 (2004), 43-54.

[4] Tran Dao Dong, *Some results on semisimple symmetric paces and invariant differential operators*, Hue University's Journal of Science, 116 (02)(2016), 11-18.

[5] Tran Dao Dong,*A compact imbedding of Riemannian symmetric spaces*, Hue University's Journal of Science, 127 (1A)(2018), 55-65.

[6] Lizhen Ji, *Introduction to symmetric spaces and their compactifications*, Lectures for the European School of Group Theory, Luminy, France, 2001.

[7] T. Oshima, *A realization of Riemannian symmetric spaces*, Journal Math. Soc. Japan, 30 (1978), 117-132.

[8] T. Oshima, *A realization of semisimple symmetric spaces and construction of boundry value maps*. Advanced Studies in Pure Math., 14 (1988), 603-650.

[9] T. Oshima and J. Sekiguchi, *Eigenspaces of invvariant differential operatorson an affine symmetric space*, Invent Math., 57 (1980), 1-81.

[10] N. Shimeno, *A compact imbbeding of semisimple symmetric spaces*, Journal Math. Sci. Univ. Tokyo, 3 (1996), 551-569.